Quantentheorie 2 UE 6

12) a) $F^{(1)} = \sum_{\alpha\alpha'} f^{(1)}_{\alpha'\alpha} \hat{a}^+_{\alpha'} \hat{a}_\alpha$ $\qquad f^{(1)}_{\alpha'\alpha} = \langle \varphi_{\alpha'} | \hat{f}^{(1)} | \varphi_\alpha \rangle$

$$F^{(1)} = \sum_{\beta\beta'} \sum_{\alpha\alpha'} \langle \varphi_{\alpha'} | \tilde{\varphi}_{\beta'} \rangle \underbrace{\langle \varphi_{\beta'} | f^{(1)} | \tilde{\varphi}_\beta \rangle}_{f^{(1)}_{\beta\beta'}} \langle \tilde{\varphi}_\beta | \varphi_\alpha \rangle \hat{a}^+_{\alpha'} \hat{a}_\alpha =$$

$$= \sum_{\beta\beta'} f^{(1)}_{\beta\beta'} \sum_{\alpha'} \langle \varphi_{\alpha'} | \tilde{\varphi}_{\beta'} \rangle \hat{a}^+_{\alpha'} \sum_\alpha \langle \tilde{\varphi}_\beta | \varphi_\alpha \rangle \hat{a}_\alpha = \sum_{\beta\beta'} f^{(1)}_{\beta\beta'} \hat{\tilde{a}}^+_{\beta'} \hat{a}_\beta$$

$$\hat{\tilde{a}}^+_\beta = \sum_\alpha \langle \varphi_\alpha | \tilde{\varphi}_\beta \rangle \hat{a}^+_\alpha$$

$$\hat{\tilde{a}}_\beta = \sum_\alpha \langle \tilde{\varphi}_\beta | \varphi_\alpha \rangle \hat{a}_\alpha$$

b) $[\hat{\tilde{a}}_i, \hat{\tilde{a}}^+_j] = \left[\sum_\alpha \langle \tilde{\varphi}_i | \varphi_\alpha \rangle \hat{a}_\alpha , \sum_{\alpha'} \langle \varphi_{\alpha'} | \tilde{\varphi}_j \rangle \hat{a}^+_{\alpha'} \right] =$

$$= \sum_{\alpha\alpha'} \langle \tilde{\varphi}_i | \varphi_\alpha \rangle \langle \varphi_{\alpha'} | \tilde{\varphi}_j \rangle \underbrace{[\hat{a}_\alpha, \hat{a}^+_{\alpha'}]}_{\delta_{\alpha\alpha'}} = \sum_\alpha \langle \tilde{\varphi}_i | \varphi_\alpha \rangle \langle \varphi_\alpha | \tilde{\varphi}_j \rangle = \delta_{ij}$$

$$[\hat{\tilde{a}}_i, \hat{\tilde{a}}_j] = \sum_{\alpha\alpha'} \langle \tilde{\varphi}_i | \varphi_\alpha \rangle \langle \tilde{\varphi}_j | \varphi_{\alpha'} \rangle \underbrace{[\hat{a}_\alpha, \hat{a}_{\alpha'}]}_{=0} = 0$$

c) $\hat{H} = -t \sum_{i=1}^{M} \sum_{j=i\pm1} \hat{a}^+_i \hat{a}_j$

$$\hat{\tilde{a}}^+_{k_n} = \frac{1}{\sqrt{M}} \sum_i e^{-ik_n x_i} \hat{a}^+_i$$

$$\hat{a}^+_i = \frac{1}{\sqrt{M}} \sum_n e^{ik_n x_i} \hat{\tilde{a}}^+_{k_n}$$

$$\hat{\tilde{a}}_{k_n} = \frac{1}{\sqrt{M}} \sum_i e^{ik_n x_i} \hat{a}_i$$

$$\hat{a}_i = \frac{1}{\sqrt{M}} \sum_n e^{-ik_n x_i} \hat{\tilde{a}}_{k_n}$$

$$\hat{H} = -\frac{t}{M} \sum_{i=1}^{M} \sum_{j=i\pm1} \sum_{nn'} e^{ik_n x_i - ik_{n'} x_j} \hat{\tilde{a}}^+_{k_n} \hat{\tilde{a}}_{k_{n'}} =$$

$$= -\frac{t}{M} \sum_{i=1}^{M} \sum_{nn'} e^{i(k_n - k_{n'}) x_i} \left(e^{ik_n \ell} + e^{-ik_{n'} \ell} \right) \hat{\tilde{a}}^+_{k_n} \hat{\tilde{a}}_{k_{n'}} =$$

$$= \left| \delta_{nn'} = \frac{1}{M} \sum_{i=1}^{M} e^{i(k_n - k_{n'}) x_i} \right| = -2t \sum_n \cos(k_n \ell) \hat{\tilde{a}}^+_{k_n} \hat{\tilde{a}}_{k_n}$$

Randbedingungen: $e^{ik x_0} = e^{ik M\ell} = 1$

$$k_n = \frac{2\pi}{M\ell} n \qquad n \in \left(-\frac{M}{2}, \frac{M}{2} \right] \ \ldots \text{ 1. Brillouin-Zone}$$

$$E_i = -2t \cos\left(\frac{2\pi}{M} i \right) \hat{n}_i \qquad i \in \left[0, \frac{M}{2} \right]$$

für $i = 0$, $i = \frac{M}{2}$ nicht entartet, sonst 2-fach entartet

Damit die verwendete Transformation tatsächlich der Rücktransformation entspricht muss gelten:

$$\hat{a}^{+}_{k_n} = \frac{1}{\sqrt{M}} \sum_i e^{-ik_n x_i} \hat{a}^{+}_i = \frac{1}{M} \sum_i e^{-ik_n x_i} \sum_m e^{ik_m x_i} \hat{a}^{+}_{k_m} =$$

$$= \frac{1}{M} \sum_m \underbrace{\sum_i e^{-i(k_n - k_m) x_i}}_{\overset{!}{=} M\delta_{nm}} \hat{a}^{+}_{k_m} \overset{!}{=} \hat{a}^{+}_{k_m}$$

$$\delta_{nm} \overset{!}{=} \frac{1}{M} \sum_{j=1}^{M} e^{i(k_n - k_m)\overbrace{jl}^{x_j}} = \frac{1 - \exp(iMl(k_n - k_m))}{\exp(-il(k_n - k_m)) - 1}$$

$k_n = k_m$ de l'Hospital $\rightarrow \frac{iMl}{il} = M$

$k_n \neq k_m$ $1 - \exp(iMl(k_n - k_m)) \overset{!}{=} 0$

$$k_n - k_m = \frac{2\pi n}{lM} \qquad n \in \mathbb{Z}$$

$\Rightarrow$ $k_n = \frac{2\pi n}{lM}$ M unterscheidbare k-Vektoren

d) Für $M=4$ Gitterplätze

$k_n \in \{0, \pm\frac{\pi}{2}, \pi\}$ $\quad E_0 = -2t \quad E_1 = 0 \quad E_2 = 2t$

$N = 1$ Teilchen $\quad E_0 = -2t \quad E_1 = 0$

1. Quantisierung $\quad |E_0\rangle_{N=1} = |k=0\rangle \quad |E_1\rangle_{N=1} = \frac{1}{\sqrt{2}}(\alpha|k=\frac{\pi}{2}\rangle + \beta|k=-\frac{\pi}{2}\rangle)$

2. Quantisierung $\quad |E_0\rangle_{N=1} = \hat{a}^+_{k=0}|vac\rangle \quad |E_1\rangle_{N=1} = \frac{1}{\sqrt{2}}(\alpha\hat{a}^+_{k=\frac{\pi}{2}} + \beta\hat{a}^+_{k=-\frac{\pi}{2}})|vac\rangle$

$N = 4$ Teilchen $\quad E_0 = -8t \quad E_1 = -6t$

1. Quantisierung $\quad |E_0\rangle_{N=4} = \prod_{i=1}^{4}|k=0\rangle_i \quad$ i,j ... Teilchenindizes

$$|E_1\rangle_{N=4} = \frac{1}{\sqrt{8}}\sum_{i=1}^{4}\prod_{j\neq i}(\alpha|k=\frac{\pi}{2}\rangle_i + \beta|k=-\frac{\pi}{2}\rangle_i)|k=0\rangle_j$$

2. Quantisierung $\quad |E_0\rangle_{N=4} = \frac{1}{\sqrt{4!}}(\hat{a}^+_{k=0})^4|vac\rangle \quad |\alpha|^2 + |\beta|^2 = 1$

$$|E_1\rangle_{N=4} = \frac{1}{\sqrt{2}}\left(\frac{\alpha}{\sqrt{3!}}\hat{a}^+_{k=\frac{\pi}{2}}(\hat{a}^+_{k=0})^3 + \frac{\beta}{\sqrt{3!}}\hat{a}^+_{k=-\frac{\pi}{2}}(\hat{a}^+_{k=0})^3\right)|vac\rangle$$

$$\hat{H} = \frac{U}{2}\sum_{i=1}^{M}\hat{a}^+_i\hat{a}^+_i\hat{a}_i\hat{a}_i = \frac{U}{2}\sum_{i=1}^{M}\hat{a}^+_i(\hat{a}_i\hat{a}^+_i - 1)\hat{a}_i = \frac{U}{2}\sum_{i=1}^{M}n_i(n_i-1)$$

$N = 1$ Teilchen $\quad E = 0 \quad$ 4-fach entartet

$N = 4$ Teilchen $\quad E = 0 \quad n_i = 1 \ \forall i \quad$ keine Entartung

Quantentheorie 2 UE 6

13) Jaynes-Cummings-Modell (in der rotating-wave Näherung

$$H = \sum_{i=0,1} E_i \hat{c}_i^\dagger \hat{c}_i + \hbar\omega_r \hat{a}^\dagger \hat{a} + \hbar g (\hat{a}^\dagger \hat{c}_0^\dagger \hat{c}_1 + \hat{c}_1^\dagger \hat{c}_0 \hat{a})$$

sonst $H_{int} = \hbar g(\hat{a} + \hat{a}^\dagger)(\hat{\sigma}_+ + \hat{\sigma}_-)$ gültig in der Nähe der Resonanz)

a) Kein externes Feld, d.h. keine Photonen im Grundzustand

$\psi_0 = c_0^\dagger |vac\rangle$ $\qquad E_0 = 0$

b) $|1\rangle = \frac{1}{\sqrt{n!}} \hat{c}_1^\dagger (\hat{a}^\dagger)^n |vac\rangle \qquad |0\rangle = \frac{1}{\sqrt{(n+1)!}} \hat{c}_0^\dagger (\hat{a}^\dagger)^{n+1} |vac\rangle$

$$\begin{array}{ccc} |1\rangle & |0\rangle & \\ \hbar\omega_a + \hbar\omega_r n & \sqrt{n+1}\,\hbar g & |1\rangle \\ \sqrt{n+1}\,\hbar g & \hbar\omega_r (n+1) & |0\rangle \end{array}$$

$$E_\pm = \hbar\left(\frac{\omega_r + \omega_a}{2} + n\omega_r\right) \pm \frac{\hbar}{2}\sqrt{4(n+1)g^2 + (\omega_r - \omega_a)^2}$$

$\delta = \omega_a - \omega_r \qquad \Omega_n = 2g\sqrt{n+1} \qquad \Omega_n(\delta) = \sqrt{\delta^2 + \Omega_n^2}$

Eigenzustände:

E_-: $\left(\hbar \frac{\omega_a - \omega_r}{2} + \frac{\hbar}{2}\Omega_n(\delta)\right) v_1 + \sqrt{n+1}\, g\, v_2 = 0$

$\sqrt{n+1}\,\hbar g\, v_1 + \left(\hbar \frac{\omega_r - \omega_a}{2} + \frac{\hbar}{2}\Omega_n(\delta)\right) v_2 = 0$

$$v_1 = \frac{1}{2g\sqrt{n+1}}\left((\omega_r - \omega_a) + \Omega_n(\delta)\right) v_2 = \frac{1}{\Omega_n}\left(\Omega_n(\delta) - \delta\right) v_2$$

$V_- \sim \begin{pmatrix} \Omega_n(\delta) - \delta \\ \Omega_n \end{pmatrix}$ $\qquad$ mit $\tan\vartheta = \frac{\Omega_n}{\Omega_n(\delta) - \delta}$

$|E_-\rangle = \cos(\vartheta)\,|1\rangle + \sin(\vartheta)\,|0\rangle$

$|E_+\rangle = -\sin(\vartheta)\,|1\rangle + \cos(\vartheta)\,|0\rangle$

c) Zweite quantisierung: $|E_-\rangle = \left(\frac{\cos(\vartheta)}{\sqrt{n!}}\hat{c}_1^+(\hat{a}^+)^n + \frac{\sin(\vartheta)}{\sqrt{(n+1)!}}\hat{c}_0^+(\hat{a}^+)^{n+1}\right)|vac\rangle$

$$|E_+\rangle = \left(-\frac{\sin(\vartheta)}{\sqrt{n!}}\hat{c}_1^+(\hat{a}^+)^n + \frac{\cos(\vartheta)}{\sqrt{(n+1)!}}\hat{c}_0^+(\hat{a}^+)^{n+1}\right)|vac\rangle$$

Darstellung der Basisfunktionen in "erster" Quantisierung (= explizite Teilchenindizes)

$|1\rangle = |E_1\rangle \prod_{i=1}^{n} |\psi_{photon}\rangle_i \qquad |0\rangle = |E_0\rangle \prod_{i=1}^{n+1} |\psi_{photon}\rangle_i$

d) $|\psi(t=0)\rangle = |1\rangle = \cos(\vartheta)|E_-\rangle - \sin(\vartheta)|E_+\rangle$

$$|\psi(t)\rangle = \exp\left[-i\left(\frac{\omega_r+\omega_m}{2}+n\omega_r\right)t\right]\Big(\exp\left[+\tfrac{i}{2}\Omega_n(\delta)t\right]\cos(\vartheta)|E_-\rangle -$$
$$-\exp\left[-\tfrac{i}{2}\Omega_n(\delta)t\right]\sin(\vartheta)|E_+\rangle\Big) =$$

$$= \exp\left[-i\left(\frac{\omega_r-\omega_m}{2}+n\omega_r\right)t\right]\Big(\exp\left[+\tfrac{i}{2}\Omega_n(\delta)t\right]\left(\cos(\vartheta)^2|1\rangle + \cos(\vartheta)\sin(\vartheta)|0\rangle\right) +$$
$$\exp\left[-\tfrac{i}{2}\Omega_n(\delta)t\right]\left(\sin(\vartheta)^2|1\rangle - \cos(\vartheta)\sin(\vartheta)|0\rangle\right)\Big)$$

$$|\langle 0|\psi(t)\rangle|^2 = \left|\left(\exp\left[+\tfrac{i}{2}\Omega_n(\delta)t\right] - \exp\left[-\tfrac{i}{2}\Omega_n(\delta)t\right]\right)\underbrace{\cos(\vartheta)\sin(\vartheta)}_{\frac{1}{2}\sin(2\vartheta)}\right|^2 =$$
$$= \sin^2\left(\tfrac{1}{2}\Omega_n(\delta)t\right)\sin^2(2\vartheta)$$

Damit Wahrscheinlichkeit 1 werden kann muss

$\sin^2(2\vartheta) = 1 \implies \vartheta \overset{!}{=} \frac{\pi}{4}$

$\tan(\vartheta^!) = 1 \implies \frac{\Omega_n}{\Omega_n(\delta)-\delta} = 1 \implies \delta = 0$