Quantentheorie 2 — 1. Tutorium 17.10.14

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1) a) $H = -g_s \mu_B \vec{S} \cdot \vec{B} \qquad \vec{S} = \frac{\hbar}{2}\vec{\sigma} \qquad \vec{B} = (0, B_y, B_z)$

$$= -\frac{g_s \mu_B \hbar}{2}(B_z \sigma_z + B_y \sigma_y)$$

In σ_z-Basis: $\sigma_z = \begin{pmatrix} 1 & \\ & -1 \end{pmatrix} \qquad \sigma_y = \begin{pmatrix} & -i \\ i & \end{pmatrix} \qquad \sigma_x = \begin{pmatrix} & 1 \\ 1 & \end{pmatrix}$

$$H = H_0 + V = -\frac{g_s \mu_B \hbar}{2} B_z \begin{pmatrix} 1 & \\ & -1 \end{pmatrix} - \frac{g_s \mu_B \hbar}{2} B_y \begin{pmatrix} & -i \\ i & \end{pmatrix} =$$

$$= -\frac{g_s \mu_B \hbar}{2}\begin{pmatrix} B_z & -iB_y \\ iB_y & -B_z \end{pmatrix}$$

b) $|\psi(t=0)\rangle = |S_z = -\frac{1}{2}\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \qquad B_y = \frac{3}{4} B_z$

i) $|\psi_s(t)\rangle = U(t)|\psi_s(0)\rangle = e^{-\frac{i}{\hbar}Ht}|\psi_s(0)\rangle = \sum_{n=1}^{2} e^{-\frac{i}{\hbar}E_n t}|n\rangle\langle n|\psi_s(0)\rangle$

→ Eigenwerte u Eigenvektoren von H finden

$$H = -\frac{g_s \mu_B \hbar}{2} B_z \begin{pmatrix} 1 & -i3/4 \\ i3/4 & -1 \end{pmatrix}$$

$\lambda_1 = \frac{5}{4}\left(\frac{g_s \mu_B \hbar}{2} B_z\right) \quad |1\rangle = \frac{1}{\sqrt{10}}\begin{pmatrix} i \\ 3 \end{pmatrix} \qquad \langle 1|\psi(0)\rangle = \frac{3}{\sqrt{10}}$

$\lambda_2 = -\frac{5}{4}\left(\frac{g_s \mu_B \hbar}{2} B_z\right) \quad |2\rangle = \frac{1}{\sqrt{10}}\begin{pmatrix} -i3 \\ 1 \end{pmatrix} \qquad \langle 2|\psi(0)\rangle = \frac{1}{\sqrt{10}}$

$$|\psi_s(t)\rangle = \frac{3}{10}\exp\left(-i\underbrace{\tfrac{5}{8} g_s \mu_B B_z t}_{\omega}\right)\begin{pmatrix} i \\ 3 \end{pmatrix} + \frac{1}{10}\exp\left(i\underbrace{\tfrac{5}{8} g_s \mu_B B_z t}_{\omega}\right)\begin{pmatrix} -i3 \\ 1 \end{pmatrix} =$$

$$= \frac{1}{10}\begin{pmatrix} 6\sin(\omega t) \\ 10\cos(\omega t) - i8\sin(\omega t) \end{pmatrix}$$

ii) $|\psi_H(t)\rangle = |\psi_H\rangle = |\psi_s(0)\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

iii) $|\psi_I(t)\rangle = U_0^+ |\psi_S(t)\rangle = \sum_{n_0=1}^{2} e^{\frac{i}{\hbar}H_0 t} |n_0\rangle\langle n_0|\psi_S(t)\rangle$ (2)

Eigenvektoren von H_0 : $|1_0\rangle = \begin{pmatrix}1\\0\end{pmatrix}$ $E_{0_1} = -\frac{g_s \mu_B \hbar}{2} B_z$

$|2_0\rangle = \begin{pmatrix}0\\1\end{pmatrix}$ $E_{0_2} = \frac{g_s \mu_B \hbar}{2} B_z$

$\langle 1_0|\psi_S(t)\rangle = \frac{6}{10}\sin(\omega t)$

$\langle 2_0|\psi_S(t)\rangle = \frac{1}{10}(10\cos(\omega t) - i\, 8 \sin(\omega t))$

$$|\psi_I(t)\rangle = \frac{1}{10}\exp\left(i\frac{g_s\mu_B}{2}B_z t\right)\begin{pmatrix}6\sin(\omega t)\exp(-i g_s \mu_B B_z t)\\ 10\cos(\omega t) - i\, 8\sin(\omega t)\end{pmatrix}$$

c) $\langle\psi_S(t)|S_z|\psi_S(t)\rangle = \frac{\hbar}{2}\frac{1}{100}(36\sin^2(At) - 100\cos^2(At) - 64\sin^2(At)) =$

$= -\frac{\hbar}{2}\frac{1}{100}(100\cos^2(\omega t) + 28\sin^2(\omega t)) = -\frac{\hbar}{2}\left(\frac{16}{25} + \frac{9}{25}\cos(2\omega t)\right)$

Man sieht sofort, dass sich die Exponentialfunktionen in $|\psi_I(t)\rangle$ und $\langle\psi_I(t)|$ aufheben und dass der Erwartungswert mit dem im Schrödinger-Bild ident ist da $(S_z)_I = U_0^+ (S_z)_S U_0 = (S_z)_S$

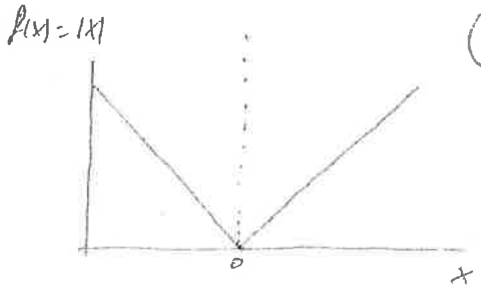
2) a) $\|\psi^2(x)\| = \int_{-\infty}^{\infty} dx\, A^2 \underbrace{(a+|x|)^{-2\nu}}_{\text{gerade Funktion}} = 2\int_0^{\infty} dx\, A^2 (a+x)^{-2\nu} =$

$\left| a + x \to y \right| = 2\int_a^{\infty} dy\, A^2 y^{-2\nu} = -\frac{2A^2}{1-2\nu} a^{1-2\nu} \overset{!}{=} 1$

$\Rightarrow A = \sqrt{\frac{2\nu - 1}{2a^{1-2\nu}}}$

2) b) $\frac{\partial}{\partial x}|x| = \mathrm{sign}(x) = \Theta(x) - \Theta(-x)$

$\frac{\partial^2}{\partial x^2}|x| = \delta(x) - (-1)\delta(-x) = 2\delta(x)$



$$\langle E \rangle = \int_{-\infty}^{\infty} \psi^*(x) H(x) \psi(x) = A^2 \int_{-\infty}^{\infty} dx \, (a + |x|)^{-\nu} \left(-\frac{\hbar^2}{2m} \frac{d^2}{dx^2} - \alpha\delta(x) \right) (a + |x|)^{-\nu} =$$

$\ulcorner$ NR: $\frac{d^2}{dx^2}(a + |x|)^{-\nu} = \frac{d}{dx}(-\nu)(a + |x|)^{-\nu-1}(\Theta(x) - \Theta(-x)) =$

$$= (1+\nu)\nu\,(a + |x|)^{-2-\nu}\underbrace{(\Theta(x) - \Theta(-x))^2}_{=1} - 2\nu(a + |x|)^{-1-\nu}\delta(x) \lrcorner$$

$$= A^2 \int_{-\infty}^{\infty} dx \left(\frac{\hbar^2 \nu}{m}(a + |x|)^{-1-2\nu}\delta(x) - \frac{\hbar^2}{2m}(1+\nu)\nu(a + |x|)^{-2-2\nu} \right) - A^2 \alpha a^{-2\nu} =$$

$\ulcorner$ NR: $\int_{-\infty}^{\infty} dx \, (a + |x|)^{-2-2\nu} = \frac{2a^{-1-2\nu}}{1+2\nu}$ $\lrcorner$

$$= A^2 \left(\nu - \frac{\nu + \nu^2}{1+2\nu} \right) \frac{\hbar^2}{m} a^{-1-2\nu} - \alpha a^{-2\nu}) =$$

$$= \frac{2\nu - 1}{2} \left(\frac{\nu^2}{1+2\nu} \frac{\hbar^2}{m} a^{-2} - \alpha a^{-1} \right)$$

$$\frac{\partial}{\partial a}\langle E \rangle = \frac{2\nu - 1}{2} \left(-\frac{2\nu^2}{2\nu+1} \frac{\hbar^2}{m} a^{-3} + \alpha a^{-2} \right) \stackrel{!}{=} 0$$

$$a_{min} = \frac{2\hbar^2\nu^2}{(1+2\nu)m\alpha}$$

$$\langle E \rangle \Big|_{a=a_{min}} = \frac{2\nu-1}{2} \left(\frac{(1+2\nu)m\alpha^2}{4\hbar^2\nu^2} - \frac{(1+2\nu)m\alpha^2}{2\hbar^2\nu^2} \right) = \frac{(1-4\nu^2)m\alpha^2}{8\nu^2\hbar^2}$$

c)

$$\frac{\partial}{\partial \nu}\langle E \rangle \Big|_{a=a_{min}} = \frac{-8\nu m\alpha^2}{8\nu^2\hbar^2} - \frac{(2-8\nu^2)m\alpha^2}{8\nu^3\hbar^2} = \frac{-m\alpha^2}{4\hbar^2\nu^2} \stackrel{!}{=} 0$$

Obige Gleichung ist nur für $\nu \to \infty$ erfüllt

2) c) $\lim_{\varepsilon \to 0} \int_{-\varepsilon}^{\varepsilon} dx \left(-\frac{\hbar^2}{2m} \frac{d}{dx} - \alpha \delta(x) \right) \psi(x) \overset{!}{=} 0$

$\Rightarrow \mathrm{I} \quad \psi_+'(0) - \psi_-'(0) = \frac{-2m\alpha}{\hbar^2} \psi(0)$

Stetigkeit: $\mathrm{II} \quad \psi_+(0) - \psi_-(0) = 0$ Ansatz: $\psi_\pm = e^{\pm ikx}$

$\mathrm{I} \quad 2ik = -\frac{2m\alpha}{\hbar^2} \Rightarrow k = \frac{im\alpha}{\hbar^2} \cdot i\varkappa$

$\psi = C e^{-\varkappa |x|}$ $\qquad ||\psi(x)^2|| = C^2 \int_{-\infty}^{\infty} dx\, e^{-2\varkappa |x|} = 2C^2 \int_0^{\infty} dx\, e^{-2\varkappa x} =$

$\psi = \sqrt{\varkappa}\, e^{-\varkappa |x|}$ $\qquad = \frac{C^2}{\varkappa} = 1 \Rightarrow C = \sqrt{\frac{m\alpha}{\hbar^2}}$

$\langle \underline{E} \rangle = \varkappa \int_{-\infty}^{\infty} dx\, e^{-\varkappa |x|} \left(-\frac{\hbar^2}{2m} \frac{d^2}{dx^2} - \alpha \delta(x) \right) e^{-\varkappa |x|} =$

$= \varkappa \int_{-\infty}^{\infty} dx \left(-\frac{\hbar^2 \varkappa^2}{2m} + \underbrace{\frac{\hbar^2}{m}\varkappa}_{\alpha} \delta(x) - \alpha \delta(x) \right) e^{-2\varkappa |x|} =$

$= -\frac{\hbar^2 \varkappa^2}{2m} \int_{-\infty}^{\infty} dx\, \varkappa\, e^{-2\varkappa |x|} = -\frac{\hbar^2 \varkappa^2}{2m} = -\frac{m\alpha^2}{2\hbar^2}$

Vergleich: $\lim_{v \to \infty} \langle E \rangle \Big|_{a = a_{min}} = \lim_{v \to \infty} \frac{m\alpha^2}{2\hbar^2} \left(\frac{1}{4v^2} - 1 \right) = -\frac{m\alpha^2}{2\hbar^2}$

$\lim_{v \to \infty} A (a + |x|)^{-v} = \lim_{v \to \infty} \sqrt{\frac{2v-1}{2}}\, a^{-\frac{1}{2}} \left(1 + \frac{|x|}{a} \right)^{-v} =$

$= \lim_{v \to \infty} \sqrt{\frac{2v-1}{2}} \sqrt{\frac{(1+2v) m\alpha}{2\hbar^2 v^2}} \left(1 + \frac{|x|}{a} \right)^{-v} =$

$= \lim_{v \to \infty} \sqrt{\frac{(4v^2-1) m\alpha}{4v^2 \hbar^2}} \left(1 + \frac{(1+2v) m\alpha}{2\hbar^2 v^2} |x| \right)^{-v} =$

$= \sqrt{\frac{m\alpha}{\hbar^2}} \exp\left(-\frac{m\alpha}{\hbar^2} |x| \right)$