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Black Holes I — Exercise sheet 3

(3.1) **Lorentz tensor gymnastics**
Take the following Lorentz tensor and vector

$$T_{\mu\nu} = \begin{pmatrix} 2 & 0 & 0 & -1 \\ 0 & -1 & 3 & 0 \\ 0 & 3 & 2 & 1 \\ -1 & 0 & 1 & 1 \end{pmatrix} \qquad k_\mu = \begin{pmatrix} 3 \\ 1 \\ 0 \\ -1 \end{pmatrix}$$

and calculate the following quantities

(a) $T^{\mu\nu}$ and $T^\mu{}_\mu$
(b) k^μ and $k_\mu k^\mu$ (is k time-, light- or spacelike?)
(c) $T_{(\mu\nu)} = \frac{1}{2}\,(T_{\mu\nu} + T_{\nu\mu})$ and $T_{[\mu\nu]} = \frac{1}{2}\,(T_{\mu\nu} - T_{\nu\mu})$
(d) $T_{\mu\nu}k^\nu$ and $T_{\mu\nu}k^\mu k^\nu$

(3.2) **Euler–Lagrange equations**
Vary the following actions and write down the Euler–Lagrange equations of motion:

(a) $S = -\int dt\,\big[q^i\dot{p}_i + H(q^i, p_i)\big]$
(b) $S = \int dt\,\big[k_1(q)\ddot{q} + k_2(q)\dot{q} - V(q)\big]$
(c) $S = -\frac{1}{2}\int d^nx\,\big[(\partial_i\phi)(\partial_j\phi)\eta^{ij} - m^2\phi^2 + \lambda\phi^4\big] \quad i, j = 0, 1, \dots, (n-1)$
(d) $S = \int dt\, q$

(3.3) **Minkowski metric in rotating coordinates**
Start with the Minkowski line-element

$$ds^2 = \eta_{\mu\nu}\,dx^\mu dx^\nu = -dt^2 + dx^2 + dy^2 + dz^2$$

and introduce "rotating coordinates"

$$t' = t \qquad x' = r\,\cos(\phi - \omega t) \qquad y' = r\,\sin(\phi - \omega t) \qquad z' = z$$

where $r = \sqrt{x^2 + y^2}$ and $\phi = \arctan(y/x)$. Find the components of the metric $g_{\mu\nu}$ and its inverse $g^{\mu\nu}$ in these coordinates, where

$$ds^2 = g_{\mu\nu}\,dx'^\mu dx'^\nu = \eta_{\mu\nu}\,dx^\mu dx^\nu$$

These exercises are due on November 2nd 2009.

Hints:

- Remember the Einstein summation convention, i.e., to sum over contracted indices. Indices are raised with the inverse Minkowski metric $\eta^{\mu\nu} = \mathrm{diag}\,(-1,1,1,1)^{\mu\nu}$ and lowered with the Minkowski metric $\eta_{\mu\nu} = \mathrm{diag}\,(-1,1,1,1)_{\mu\nu}$.

- You may drop all boundary terms/total derivative terms and use partial integrations whenever a derivative acts on a variation (you may also keep boundary terms, and you will have made your first step towards understanding D-branes). And yes, the answer you get for the equations of motion in the case (d) is really strange...

- Remember that $dx'^{\mu} = dx^{\nu}\,\frac{\partial x'^{\mu}}{\partial x^{\nu}}$ and insert this into the last formula to extract $g_{\mu\nu}$. You get $g^{\mu\nu}$ e.g. from taking the matrix inverse of $g_{\mu\nu}$, but this is not the only possibility.