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Black Holes I — Exercise sheet 4

(4.1) **Geodesics in weak gravity fields**
Take the metric

$$g_{\mu\nu}dx^\mu dx^\nu = -\big(1+2\Phi(r)\big)\,dt^2 + dr^2 + r^2\,d\theta^2 + r^2\sin^2\theta\,d\phi^2$$

with some arbitrary function $\Phi(r)$ and calculate geodesics in the plane $\theta = \pi/2$. Assume that velocities are small and write the geodesic equation as $m\,a = F(r)$. What kind of force do you get for $F(r)$? How are Φ and F related? What happens in the special case $\Phi = -M/r$? What happens in the special case $\Phi = M/r$? Which of these cases is physically irrelevant and why?

(4.2) **Light-bending**
Derive a formula for gravitational light-bending, starting from the Schwarzschild metric given in the last exercise. What is the difference to the Newtonian formula for light-bending?

(4.3) **Christoffel symbols for Schwarzschild metric**
Take the Schwarzschild metric

$$g_{\mu\nu}dx^\mu dx^\nu = -(1-2M/r)\,dt^2 + \frac{dr^2}{1-2M/r} + r^2\,d\theta^2 + r^2\sin^2\theta\,d\phi^2$$

and calculate all Christoffel symbols of the second kind

$$\Gamma^\alpha{}_{\beta\gamma} = \frac{1}{2}\,g^{\alpha\mu}\big(\partial_\beta g_{\mu\gamma} + \partial_\gamma g_{\mu\beta} - \partial_\mu g_{\beta\gamma}\big)$$

These exercises are due on November 16th 2009.

Hints:

- You may assume $\dot{\theta} = 0$, analog to the lectures. Then there are two routes: 1. Calculate the required Christoffel symbols of the second kind (see last exercise for a definition) and insert them into the geodesic equation we derived in the previous lectures. Note that you do *not* have to calculate all Christoffel symbols, but only a relatively small subset of them! 2. A convenient short-cut is to proceed exactly as we did in the lectures, except that you do not insert the Schwarzschild metric but its Newtonian approximation into the geodesic action.

- Calculations are simpler than for the perihelion shift, but you have to know what to calculate. You are looking for a formula for the deflection angle of a light-ray (which moves on a null geodesic) in the gravitational field of a point source (like the Sun). You can use the results we derived for null geodesics in the Schwarzschild background. Establish an integral formula for the azimuthal angle ϕ as a function of the radial coordinate r (or its inverse $1/r$). Evaluating this integral is tricky and leads to elliptic functions. Either evaluate them brute force with something like Maple or Mathematica or make a perturbative expansion, exploiting the fact that $M/r \ll 1$. Alternatively, you may use a different set of coordinates where the integral simplifies, so-called isotropic coordinates (google for "Schwarzschild metric", go to the wikipedia page and check the paragraph "Alternative (isotropic) formulations of the Schwarzschild metric"). Newtonian light-bending works in the same way, but you take the Newton approximation to the Schwarzschild metric instead of the latter, i.e., you set to unity the dr^2 coefficient in the line-element.

- This exercise is not difficult, since you just have to differentiate and sum various terms. However, it is quite some work. In fact, probably you will be bored at some point. But you really should do this calculation once in your lifetime. It helps you to appreciate the usefulness of Computer Algebra Systems and of more powerful formalisms in differential geometry.