Daniel Grumiller November 16th 2009

Black Holes I — Exercise sheet 5

(5.1) **Lie derivatives**
Given the three vector fields

$$L_0 = \partial_\phi$$
$$L_1 = -\cos\phi\,\partial_\theta + \sin\phi\cot\theta\,\partial_\phi$$
$$L_2 = \sin\phi\,\partial_\theta + \cos\phi\cot\theta\,\partial_\phi$$

calculate their Lie-brackets (commutators)

$$[L_i, L_j] = f_{ij}{}^k\, L_k$$

and determine the structure constants $f_{ij}{}^k$. (If you know about Lie algebras try to find out which Lie algebra is generated by these three vector fields). Calculate also the Lie derivatives

$$\mathcal{L}_{L_i}\left(g_{\alpha\beta}\right) = L_i^\mu \partial_\mu g_{\alpha\beta} + g_{\alpha\mu}\partial_\beta L_i^\mu + g_{\mu\beta}\partial_\alpha L_i^\mu$$

of the metric of the unit sphere,

$$ds^2 = g_{\alpha\beta}\,dx^\alpha dx^\beta = d\theta^2 + \sin^2\theta\,d\phi^2$$

along all three vector fields L_i given above. What is the meaning of your result?

(5.2) **Coordinate transformation of Christoffels**
Consider a general coordinate transformation $x' = x'(x)$ and calculate the transformation of the Christoffel symbols (of the second kind). Do they transform as a tensor?

(5.3) **Riemann-tensor calculation**
Take the line element

$$ds^2 = -dt^2 + dr^2 + r^2\,d\theta^2 + r^2\sin^2\theta\,d\phi^2$$

and calculate all Riemann-tensor components.

These exercises are due on November 23rd 2009.

Hints:

- The definition of a commutator between two vector fields is $[A, B] = AB - BA$. A simple example for illumination: Take the vector fields $\xi = x\,\partial_x + y\,\partial_y$ and $\zeta = c\,\partial_x$, with $c = \text{const}$. Then their Lie bracket is $[\xi, \zeta] = x(\partial_x c)\partial_x + y(\partial_y c)\partial_x - c((\partial_x x)\partial_x + (\partial_x y)\partial_y) = -c\,\partial_x = -\zeta$. Regarding the final question of this exercise: remember what we learned in the lectures about symmetries and Killing vectors.

- The definition of the Christoffel symbols of the second kind is

$$\Gamma^\alpha{}_{\beta\gamma} = \frac{1}{2}\,g^{\alpha\mu}\big(\partial_\beta g_{\mu\gamma} + \partial_\gamma g_{\mu\beta} - \partial_\mu g_{\beta\gamma}\big)$$

  Remember that the metric is a tensor and therefore transforms under coordinate change as any other tensor does – we have learned in the lectures how this works, so if you are not sure please consult your lecture notes.

- Either you calculate all Christoffels and insert into the definition of the Riemann-tensor, or you make a suitable coordinate transformation to a simpler set of coordinates, calculate the Riemann-tensor $R^\alpha{}_{\beta\gamma\delta}$ in these simpler coordinates and use the fact that the Riemann-tensor transforms as a $(1, 3)$ tensor.