Black Holes I — Exercise sheet 6

(6.1) **p-forms and wedge product**
First some vocabulary. Totally antisymmetric tensors T of type $(0, p)$ are also called "p-forms".

$$T = T_{[\alpha_1\alpha_2\dots\alpha_p]}\,\mathrm{d}x^{\alpha_1}\,\mathrm{d}x^{\alpha_2}\dots\mathrm{d}x^{\alpha_p}$$

A scalar field is therefore also called "0-form", a dual vector "1-form" and an antisymmetric 2-tensor "2-form". A D-form in D dimensions is called "volume-form". Define now the wedge-product between a p-form P and a q-form Q as follows:

$$(P \wedge Q)_{\mu_1\mu_2\dots\mu_{p+q}} := \frac{(p+q)!}{p!q!}\,P_{[\mu_1\dots\mu_p}Q_{\mu_{p+1}\dots\mu_{p+q}]}$$

Finally, define the "de Rahm differential" or "exterior derivative" as the 1-form $\mathrm{d} := \partial_\mu\,\mathrm{d}x^\mu$. Now show the following identities, assuming that P is a p-form and Q a q-form:

(a) $P \wedge Q = (-1)^{pq} Q \wedge P$ and thus $P \wedge P = 0$ if p is odd
(b) $\mathrm{d}^2 = 0$
(c) $P \wedge Q = 0$ if $p + q > D$
(d) $(\mathrm{d}\wedge P)_{\mu\nu} = (p+1)\partial_{[\mu_1}P_{\mu_2\dots\mu_{p+1}]}$ [Note: expressions like $\mathrm{d}\wedge P$ are often abbreviated as $\mathrm{d}P$]
(e) Show that $\epsilon_{\mu_1\dots\mu_D}\,\mathrm{d}x^{\mu_1}\wedge\dots\wedge\mathrm{d}x^{\mu_D}$ is a volume form and transforms as a tensor of type $(0, D)$.

(6.2) **Covariant derivatives of anti-symmetric tensors**
Given a dual vector field A_μ and the covariant derivative ∇_μ we can construct various 2-tensors. Consider the antisymmetric tensor

$$F_{\mu\nu} = \nabla_\mu A_\nu - \nabla_\nu A_\mu$$

and show the identity

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \tag{1}$$

In relation to the discussion of p-forms in the exercise above please note that F is a 2-form and can be written compactly as $F = \mathrm{d}A$.

(6.3) **Maxwell equations**
Assume we are in flat space. Vary the action $S = \int \mathrm{d}^4x F_{\mu\nu}F^{\mu\nu}$ with respect to the gauge field A_μ, where the field strength $F_{\mu\nu}$ is related to the latter by Eq. (1) above. The equations you get are called "inhomogeneous Maxwell equations". The homogeneous Maxwell equations can be written as the vanishing of the 3-form $\mathrm{d}F = 0$ (address why this relation must hold). Finally, split the field strength into two spatial "vectors" as follows: $E_i := F_{i0}$, $B_k := \frac{1}{2}\,\epsilon_{ijk}F_{ij}$ and write all Maxwell equations in terms of E_i and B_i. Does this look familiar?

These exercises are due on November 30th 2009.

Hints:

- In (a) and (c) just use the definitions of wedge product and of forms. In (b) use the definition of the exterior derivative and the fact that second partial derivatives commute. In (d) use all definitions. In (e) note that $\epsilon_{\mu_1 \ldots \mu_D}$ are the components of the ϵ-tensor.

- Just insert the definition of the covariant derivative acting on a dual vector and prove that the connection terms cancel.

- Remember, we are here in flat spacetime, so all Greek indices are raised and lowered with the Minkowski metric and Latin indices with the Euclidean metric (thus we do not have to raise Latin indices at all). Your result for the inhomogeneous Maxwell equations should read $\partial_\mu F^{\mu\nu} = 0$ (they are called "inhomogeneous", because the right hand side may contain a source term j^ν once we add interactions with charged matter). The homogeneous Maxwell equations are a direct consequence of Eq. (1) together with the result (6.1), (b). For the final part, if you want to bring this into a form that is probably familiar to you remember the definitions $\partial_i E_i = \operatorname{div} E$ and $\epsilon_{ijk}\partial_j E_k = (\operatorname{curl} E)_i$. Also, do not hesitate to multiply equations by convenient quantities, for instance ϵ_{ijk}. Occasionally you might want to use the identity $\epsilon_{ijk}\epsilon_{inm} = \delta_{jn}\delta_{km} - \delta_{jm}\delta_{kn}$.