Daniel Grumiller November 30th 2009

Black Holes I — Exercise sheet 7

(7.1) **Killing horizon**
Take the most general spherically symmetric metric in four spacetime dimensions and assume stationarity, i.e., the existence of a Killing vector which in adapted coordinates (t, r, θ, ϕ with their standard meaning) is given by ∂_t. Calculate the norm of this Killing vector and discuss under which conditions the Killing vector is timelike (spacelike) [lightlike]. Any locus $r =$ const. where the Killing norm vanishes is known as "Killing horizon". How many Killing horizons can exist for a given metric? Apply your results to the Schwarzschild metric, show that there is exactly one Killing horizon and find in particular the radius of the Killing horizon in terms of the mass of the black hole.

(7.2) **Killing observer**
Take the Schwarzschild metric and place observers in the spacetime such that their feet touch the Schwarzschild radius and their head is $1.62m$ away from the Schwarzschild radius. Calculate the force difference per mass (a.k.a. relative acceleration) between head and feet that stretches the observer for (a) a 10 solar mass black hole, (b) a supermassive galactic black hole with 10^7 solar masses, (c) a black hole with the mass of the observable Universe. Who will survive and who will get killed? [You may assume that the observers can survive maximally up to "thousand g".] Compare with the relative acceleration that would act on the same observers on the surface of the Earth.

(7.3) **Observer horizon**
At extremely large scales today or in the inflationary phase our Universe is described well by the deSitter metric

$$\mathrm{d}s^2 = -\Big(1 - \Lambda\, r^2\Big)\,\mathrm{d}t^2 + \frac{\mathrm{d}r^2}{\Big(1 - \Lambda\, r^2\Big)} + r^2\,\big(\mathrm{d}\theta^2 + \sin^2\theta\ \mathrm{d}\phi^2\big)$$

where the observer is located at the origin $r = 0$. Calculate the position of the Killing horizon in terms of the cosmological constant Λ, which for this metric is known as "observer horizon", and determine the value of the radius of this horizon for our present Universe. Another way to obtain an observer horizon is by constantly accelerating an observer, as you have learned in special relativity. A geometric way to describe this is the Rindler metric, which in 1+1 dimensions is given by

$$\mathrm{d}s^2 = -(\kappa x)^2\,\mathrm{d}t^2 + \mathrm{d}x^2$$

where $t \in (-\infty, \infty)$ and $x \in (0, \infty)$. This metric has a Killing horizon, but also a coordinate singularity. Perform the coordinate transformation $T - X = -xe^{-\kappa t}$, $T + X = xe^{\kappa t}$ and check what kind of metric you obtain. Is the coordinate singularity a physical singularity? Discuss in the coordinates T and X (or $U = T - X$, $V = T + X$) where the original Rindler metric is defined and where the Killing horizon is located.

These exercises are due on December 7th 2009.

Hints:

- The most general spherically symmetric metric in adapted coordinates is of the form

$$\mathrm{d}s^2 = g_{tt}\,\mathrm{d}t^2 + 2g_{tr}\,\mathrm{d}t\,\mathrm{d}r + g_{rr}\,\mathrm{d}r^2 + X\left(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2\right)$$

  where all functions $g_{\alpha\beta}, X$ depend on t and r only. Assuming stationarity eliminates the t-dependence. Now you just have to calculate the norm of the vector ∂_t. Application to Schwarzschild is straightforward, once you remember that $g_{tt} = -1 + 2M/r$ for Schwarzschild.

- Remember, the Schwarzschild radius is located at $r = 2M$. For your convenience, here is the conversion of various quantities into natural units: $1.62m \approx 10^{35}$, 10 solar masses $\approx 10^{39}$, for a supermassive black hole you can use $M \approx 10^{45}$ and the mass of the observable Universe is about $M \approx 10^{61}$. To calculate the force difference per mass you might use the Newton approximation — but convince yourself whether you are allowed to do that. If you do not know what "thousand g" means look for "g-force" at wikipedia.

- The solution to this exercise probably is shorter than its description. In natural units we have $\Lambda \approx 10^{-123}$. For comparison, the radius of the visible Universe today is about $10^{61} - 10^{62}$. The second part with the Rindler metric is very simple. First deduce where the Killing horizon of the Killing vector ∂_t is located, essentially by looking at the line-element. Then just perform the coordinate transformation given in the text. To address the last issue a picture is probably helpful: draw the lines $x = 0$ in the new coordinates T, X (or U, V) and remember the range of definition of x.