Black Holes I — Exercise sheet 8

(8.1) **Much ado about nothing**

In this exercise we study the vacuum in two spacetime dimensions. Consider 2-dimensional Minkowski spacetime

$$\mathrm{d}s^2 = -\,\mathrm{d}t^2 + \mathrm{d}x^2$$

Try to find as many Killing vectors as possible and discuss what kind of symmetries they generate. Write down the algebra generated by the Lie brackets between the Killing vectors. Now consider 2-dimensional Anti-deSitter (AdS) spacetime in the so-called Poincaré patch,

$$\mathrm{d}s^2 = \frac{-\,\mathrm{d}t^2 + \mathrm{d}x^2}{x^2}$$

and find again as many Killing vectors as possible. Also, write down the algebra generated by the Lie brackets between the Killing vectors.
A victory is twice itself when the achiever brings home full numbers.

(8.2) **Measure for measure**

You have learned that a measure for the acceleration of a static particle near a Killing horizon as measured at spatial infinity is given by surface gravity κ,

$$\kappa^2 = -\frac{1}{2}\,(\nabla_\mu \xi_\nu)(\nabla^\mu \xi^\nu)\big|_{\mathcal{H}}$$

where ξ is the Killing vector that becomes null at the horizon $\mathcal{H}$. Consider a static spherically symmetric line-element

$$\mathrm{d}s^2 = -K(r)\;\mathrm{d}t^2 + \frac{\mathrm{d}r^2}{K(r)} + r^2\left(\mathrm{d}\theta^2 + \sin^2\theta\;\mathrm{d}\phi^2\right)$$

where the function $K(r)$ is chosen such that it has at least one zero in the range of definition of r. Then the Killing vector $\xi = \partial_t$ has at least one Killing horizon $\mathcal{H}$. Show that surface gravity is given by $\kappa = \frac{1}{2}\frac{\mathrm{d}K}{\mathrm{d}r}\big|_{\mathcal{H}}$. Check what you get for surface gravity of the Schwarzschild black hole. When does surface gravity vanish (in general and for Schwarzschild)?
If any in Vienna be of worth to undergo such ample grace and honour, ...

(8.3) **The comedy of errors**

Let us perform a Gedankenexperiment.[1] Suppose you have a cup of hot tea and a cup of cold tea. Now mix them, thereby irreversibly increasing the entropy of the Universe. Would it not be possible to erase all traces of your "crime" by dropping the cup of lukewarm tea into a black hole? In other words, is it not possible to work against the second law of thermodynamics by dropping highly entropic objects into a black hole? How can it be that information apparently is lost in a black hole? Does this not contradict basic results you learned in thermodynamics and quantum mechanics?
That's a question: how shall we try it?

These exercises are due on December 14th 2009.

[1] This Gedankenexperiment goes back to John Wheeler and Jacob Bekenstein.

Hints:

- Recall the definition of a Killing vector ξ,

$$\mathcal{L}_\xi g_{\mu\nu} = \xi^\alpha \partial_\alpha g_{\mu\nu} + g_{\mu\alpha}\partial_\nu \xi^\alpha + g_{\alpha\nu}\partial_\mu \xi^\alpha = 0$$

  and write down all three independent components of this equation explicitly. This establishes three linear PDEs for the two components ξ^t and ξ^x, which allow for several linearly independent solutions. All of them are reasonably simple to find. Of course, strictly speaking the number of Killing vectors you will find is infinite, because for any finite set of Killing vectors $\xi^{(i)}$ also $\sum_i a_{(i)}\,\xi^{(i)}$ is a Killing vector, with some arbitrary constants $a_{(i)}$. But you can always find a finite dimensional basis that generates all Killing vector upon linearly combining them. So when asking "how many Killing vectors are there?" we are really interested in answering the question "how many basis Killing vectors are there?" [regarding Lie bracket check exercise (5.1) in case of doubt]
  Were you in doubt, sir, that you asked her?

- Sorry, no hint this time.
  Since I am put to know that your own science
  Exceeds, in that, the lists of all advice
  My strength can give you: then no more remains,
  But that to your sufficiency as your Worth is able,
  And let them work.

- These are really tough questions, and some of the brightest minds among us physicists were led astray in attempts to answer them. So I don't expect you to come up with "the" solution to these questions, but it is still very gratifying to think about them. Just write down your thoughts and don't worry if they are likely to be wrong.
  Yet this my comfort: when your words are done,
  My woes end likewise with the evening sun.