Black Holes I — Exercise sheet 9

(9.1) **Reissner–Nordström black hole and Robinson–Bertotti limit**
Charged black holes with mass M and charge Q in four spacetime dimensions are described by the Reissner–Nordström (RN) metric

$$\mathrm{d}s^2 = -K(r)\,\mathrm{d}t^2 + \frac{\mathrm{d}r^2}{K(r)} + r^2\,\mathrm{d}\Omega^2 \qquad K(r) = 1 - \frac{2M}{r} + \frac{Q^2}{r^2}$$

where $\mathrm{d}\Omega^2 = \mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2$.
(a) Can the RN metric emerge as vacuum solution in general relativity? Why/why not?
(b) How many Killing horizons do exist for various values of M and Q? Calculate their surface area $A = 4\pi r^2$.
(c) For the case where at least one Killing horizon exists evaluate surface gravity on the largest horizon. When does surface gravity vanish?
(d) In the extremal case $M^2 = Q^2$ take the near horizon limit of the RN geometry and discuss what kind of geometry you obtain.

(9.2) **Vaidya metric**
The formation of black holes is a very complicated process. A simple toy model assumes that there is only infalling matter and no outgoing matter or radiation. Such scenarios are described by chiral matter and chiral geometries, known as “Vaidya metric”

$$\mathrm{d}s^2 = 2\,\mathrm{d}r\,\mathrm{d}v - K(r,v)\,\mathrm{d}v^2 + r^2\,\mathrm{d}\Omega^2 \qquad K(r,v) = 1 - \frac{2m(v)}{r}$$

where $m(v)$ is a function of the light-like coordinate $v \in (-\infty, \infty)$.
(a) Discuss physical problems that can arise when $m(v)$ is not everywhere positive.
(b) Discuss physical problems that can arise when $m(v)$ is not monotonically increasing with v.
(c) Consider $m(v) = M\,\theta(v - v_0)$ with some constant M (θ is the step-function). Describe how the black hole is formed in this case: how does spacetime look before $v < v_0$ and how after $v > v_0$? What does this tell you about the infalling matter?
(d) In scenario (c) suppose that you are an observer at $0 < r < 2M$, starting at $v = v_1 < v_0$. Can you escape to infinity or are you trapped behind a black hole horizon?

(9.3) **Euclidean Schwarzschild spacetime and Hawking temperature**
Perform a Wick rotation $t \to i\tau$ on the Schwarzschild spacetime. Take the near horizon limit. Drop the 2-sphere part of the line-element and focus on the 2-dimensional Euclidean geometry that remains. Under which conditions is that geometry flat Euclidean space? What kind of geometry do you obtain if the conditions are not met?
Christmas bonus: In quantum field theory typically the periodicity in Euclidean time is associated with a characteristic temperature. Assume that this holds also in the present case and calculate this temperature. Congratulations, you have just “derived” the Hawking-temperature. Merry X-mas!

These exercises are due on January 11th 2010.

Hints:

- Concerning (d): use the reparameterization $r = M(1+\lambda)$ and expand in powers of λ keeping only the leading term in each expression. The final result should be the Robinson–Bertotti metric, a direct product of two maximally symmetric 2-dimensional spacetimes, one with constant positive curvature and one with constant negative curvature — see also exercises (8.1) and (8.2).

- Regarding (d): it may seem strange that being in flat spacetime can nevertheless imply that you are hidden behind a horizon, but you can easily understand this with our venerable fishy river analogy: suppose that the river stands still initially, but at some point the current is switched on. Then some fish in the river will be trapped behind the point of no return, while others can escape, even though initially all fish experiences the same local "geometry".

- There is not much that you have to calculate that we didn't calculate already, but a few things you should think through. It is particularly useful to remind you how a cone is constructed: take flat Euclidean (2-dimensional) space and remove a wedge, thereby creating a deficit angle γ. Equivalently, you can demand that, in polar coordinates, the azimuthal angle does not have periodicity 2π but rather $2\pi - \gamma$, again creating a deficit angle γ. Note that a cone is not the same as Euclidean space: even though it is intrinsically flat, it does have a singularity at the origin, usually referred to as "conical singularity".