Daniel Grumiller November 14^{th} 2011

Black Holes I — Exercise sheet 5

(5.1) **Eddington-Finkelstein gauge**
Take the Schwarzschild line-element in Schwarzschild gauge, viz.

$$\mathrm{d}s^2 = -\Big(1-\frac{2M}{r}\Big)\,\mathrm{d}t^2 + \frac{\mathrm{d}r^2}{\Big(1-\frac{2M}{r}\Big)} + r^2\big(\,\mathrm{d}\theta^2 + \sin^2\theta\,\,\mathrm{d}\phi^2\big)$$

and perform a coordinate transformation $v = t + f(r)$ with some suitable function f (which you have to determine) so that you obtain the Schwarzschild line-element in Eddington-Finkelstein gauge, viz.

$$\mathrm{d}s^2 = 2\,\mathrm{d}r\,\mathrm{d}v - \Big(1-\frac{2M}{r}\Big)\,\mathrm{d}v^2 + r^2\big(\,\mathrm{d}\theta^2 + \sin^2\theta\,\,\mathrm{d}\phi^2\big)\,.$$

Is this line-element singular at $r = 2M$?

(5.2) **Coordinate transformation of Christoffels**
Consider a general coordinate transformation $x' = x'(x)$ and calculate the transformation of the Christoffel symbols (of the second kind). Do they transform as a tensor?

(5.3) **Covariant derivative of a constant vector field**
Given a smooth manifold in $n \geq 2$ dimensions equipped with an arbitrary metric of signature $(-,+,\dots,+)$ consider a vector field v^μ, whose components in a certain coordinate system $(t, x_1, \dots, x_{n-1})$ are given by $v^t = 1$ and $v^\mu = 0$ otherwise. Thus, in these coordinates the vector field is a constant vector field and takes the simple form $v^\mu = (1, 0, \dots, 0)$. The main task of this exercise is to calculate the covariant derivative of this vector field, $\nabla_\nu v^\mu$. (When) is it true that $\nabla_\nu v^\mu = 0$ for this vector field v^μ?

These exercises are due on November 21^{st} 2011.

Hints:

- This is perhaps the most efficient way to proceed: use the Ansatz $v = t + f(r)$, derive the coordinate differential $\mathrm{d}v$ and calculate backwards by starting with the line-element in Eddington-Finkelstein gauge and ending with the line-element in Schwarzschild gauge. Concerning the last question: the metric obviously is finite at $r = 2M$; the only issue not completely obvious is whether it is invertible at $r = 2M$.

- The definition of the Christoffel symbols of the second kind is

$$\Gamma^{\alpha}{}_{\beta\gamma} = \frac{1}{2}\, g^{\alpha\mu}\big(\partial_\beta g_{\mu\gamma} + \partial_\gamma g_{\mu\beta} - \partial_\mu g_{\beta\gamma}\big)$$

  Remember that the metric is a tensor and therefore transforms under coordinate change as any other tensor does – we have learned in the lectures how this works, so if you are not sure please consult your lecture notes.

- This is a very short exercise (it should take less than a minute to write down the answers). However, be careful before answering the question — it may be tempting to give a short but incorrect answer.