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Black Holes I — Exercise sheet 10

(10.1) **Killing horizons and surface gravity of Kerr**
Take the Kerr metric and show that the two zeros $r_+ > r_-$ of the function $\Delta = 0$ are Killing horizons of the Killing vector fields

$$\xi_\pm = \partial_t + \Omega_\pm \partial_\phi$$

where $\Omega_\pm$ are constants that you should determine. One interpretation of this result is that the event horizon (i.e., the outer Killing horizon $r = r_+$) of the Kerr black hole rotates with angular velocity Ω_+. One can prove that on the event horizon surface gravity is given by

$$\kappa = \frac{r_+ - r_-}{2(r_+^2 + a^2)}$$

Derive a condition for vanishing κ in terms of M and a.

(10.2) **Energy extraction from rotating black holes (Penrose process) and special case of Hawking's area theorem**
In the ergoregion the Killing vector $k = \partial_t$ is spacelike. Therefore, particles with 4-momentum p need not have positive energy in the ergoregion, $E = -p^a k_a < 0$. Forcing the black hole to absorb such a particle with negative energy means that we can extract positive energy from the black hole. This process is called Penrose process. From the causal inequality at the outer Killing horizon $p_a \xi_+^a \leq 0$ [where ξ_+^a is defined in exercise (10.1)] derive an inequality that relates energy $E < 0$ and angular momentum L of the infalling particle. Assume that the final black hole state has mass $M + \delta M$ and angular momentum $J + \delta J$ where $\delta M = E$ and $\delta J = L$. Derive now the inequality

$$\delta\big(M^2 + \sqrt{M^4 - J^2}\big) \geq 0$$

Show also that the result above can be rewritten as

$$\delta A \geq 0 \qquad \text{with} \qquad A = 4\pi\,(r_+^2 + a^2)$$

This is a particular example of the area theorem, which states that the surface area of a black hole can never decrease.

(10.3) **Coalescing Kerr black holes**
Assume we start with two separated co-axial[1] Kerr black holes with same masses $M > 0$ and angular momenta J (up to a sign). Now we let them merge to a single black hole. One can prove that in this case no angular momentum can be radiated away, so the final state black hole must have angular momentum $J_f = 0$ (anti-parallel rotating black holes) or $J_f = 2J$ (parallel rotating black holes). Use the area theorem from exercise (10.2) to derive an upper limit for gravitational wave energy radiated away in this process for both cases (parallel/anti-parallel). Check in particular the extremal limit $J = M^2$ and the Schwarzschild limit $J = 0$. Express your result in percentage of the total initial mass $2M$.

These exercises are due on January 21th 2014.

[1]Co-axial means that the full geometry is also axi-symmetric since both black hole axes coincide.

Hints:

In all exercises use the Kerr-metric in Boyer-Lindquist coordinates

$$\mathrm{d}s^2 = -\frac{\Delta - a^2\sin^2\theta}{\Sigma}\,\mathrm{d}t^2 - \frac{4Mar\sin^2\theta}{\Sigma}\,\mathrm{d}t\,\mathrm{d}\phi + \frac{(r^2+a^2)^2 - \Delta a^2\sin^2\theta}{\Sigma}\,\sin^2\theta\,\mathrm{d}\phi^2 + \frac{\Sigma}{\Delta}\,\mathrm{d}r^2 + \Sigma\,\mathrm{d}\theta^2\,,$$

where

$$\Sigma := r^2 + a^2\cos^2\theta \qquad \Delta := r^2(1 - \frac{2M}{r}) + a^2\,,$$

M is the black hole mass and $a = J/M = \text{const.}$ is the Kerr parameter. Here are additional hints:

- This is a very short exercise. If you are ambitious you may additionally *derive* the result for surface gravity, but this is a slightly lengthy calculation.
- If you did not do exercise (10.1) here is the Killing vector ξ_+

  $$\xi_+ = \partial_t + \Omega_+\partial_\phi \qquad \Omega_+ = \frac{a}{r_+^2 + a^2}$$

  and the locus of the event horizon:

  $$r_+ = M + \sqrt{M^2 - a^2}$$

- Exploit the fact that the area $A_f = 4\pi\,(r_{+\,f}^2 + a_f^2)$ of the final black hole state must be larger or equal to the sum of the areas of the two initial black hole states. If this inequality is saturated (i.e., it turns into an equality) you get the maximal amount of gravitational wave energy radiated away. Solve this equality for the unknown mass of the final black hole, using the relations between $r_{+\,f}$, a_f, J_f and M_f derived in the previous exercises. If you did not do these exercises, here are the relations that you need:

  $$J_f = M_f a_f \qquad r_{+\,f} = M_f + \sqrt{M_f^2 - a_f^2}$$

  Analog relations hold of course for the two initial black holes with parameters r_+, a, J and M. Note that the total energy is given by the initial energy $2M$, so the total energy radiated away is $2M - M_f$.