

Black Holes I — Exercise sheet 3

(3.1) Lorentz tensor gymnastics

Take the following Lorentz tensor and vector

$$T_{\mu\nu} = \begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & -1 & 3 & 0 \\ 0 & 3 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad k_\mu = \begin{pmatrix} -3 \\ 1 \\ 2 \\ -2 \end{pmatrix}$$

and calculate the following quantities

- (a) $T^{\mu\nu}$ and $T^\mu{}_\mu$
- (b) k^μ and $k_\mu k^\mu$ (is k time-, light- or spacelike?)
- (c) $T_{(\mu\nu)} = \frac{1}{2}(T_{\mu\nu} + T_{\nu\mu})$ and $T_{[\mu\nu]} = \frac{1}{2}(T_{\mu\nu} - T_{\nu\mu})$
- (d) $T_{\mu\nu} k^\nu$ and $T_{\mu\nu} k^\mu k^\nu$
- (e) $T_{\mu\nu} T^{\nu\lambda}$ [using the result for (a)]

(3.2) Euler–Lagrange equations

Vary the following actions and write down the Euler–Lagrange equations of motion:

- (a) $S = - \int dt [q^i \dot{p}_i + H(q^i, p_i)]$
- (b) $S = \int dt [k_1(q) \ddot{q} + k_2(q) \dot{q} - V(q)]$
- (c) $S = -\frac{1}{2} \int d^n x [(\partial_i \phi)(\partial_j \phi) \eta^{ij} - m^2 \phi^2 + \lambda \phi^4]$ $i, j = 0, 1, \dots, (n-1)$
- (d) $S = \int dt q$

(3.3) Minkowski metric in rotating coordinates

Start with the Minkowski line-element

$$ds^2 = \eta_{\mu\nu} dx^\mu dx^\nu = -dt^2 + dx^2 + dy^2 + dz^2$$

and introduce “rotating coordinates”

$$t' = t \quad x' = r \cos(\phi - \omega t) \quad y' = r \sin(\phi - \omega t) \quad z' = z$$

where $r = \sqrt{x^2 + y^2}$ and $\phi = \arctan(y/x)$. Find the components of the metric $g_{\mu\nu}$ and its inverse $g^{\mu\nu}$ in these coordinates, where

$$ds^2 = g_{\mu\nu} dx'^\mu dx'^\nu = \eta_{\mu\nu} dx^\mu dx^\nu$$

These exercises are due on November 19th 2013.

Hints:

- Remember the Einstein summation convention, i.e., to sum over contracted indices. Indices are raised with the inverse Minkowski metric $\eta^{\mu\nu} = \text{diag}(-1, 1, 1, 1)^{\mu\nu}$ and lowered with the Minkowski metric $\eta_{\mu\nu} = \text{diag}(-1, 1, 1, 1)_{\mu\nu}$.
- You may drop all boundary terms/total derivative terms and use partial integrations whenever a derivative acts on a variation (you may also keep boundary terms, and you will have made your first step towards understanding D-branes). And yes, the answer you get for the equations of motion in the case (d) is really strange...
- Remember that $dx'^{\mu} = dx^{\nu} \frac{\partial x'^{\mu}}{\partial x^{\nu}}$ and insert this into the last formula to extract $g_{\mu\nu}$. You get $g^{\mu\nu}$ e.g. from taking the matrix inverse of $g_{\mu\nu}$, but this is not the only possibility.