Daniel Grumiller November 19^{th} 2013

Black Holes I — Exercise sheet 4

(4.1) **Geodesics in weak gravity fields**
Take the metric

$$g_{\mu\nu}dx^\mu dx^\nu = -\big(1+2\Phi(r)\big)\,dt^2 + dr^2 + r^2\,d\theta^2 + r^2\sin^2\theta\,d\phi^2$$

with some arbitrary function $\Phi(r)$ and calculate geodesics in the plane $\theta = \pi/2$. Assume that velocities are small and write the geodesic equation as $m\,a = F(r)$. What kind of force do you get for $F(r)$? How are Φ and F related? What happens in the special case $\Phi = -M/r$? What happens in the special case $\Phi = M/r$? Which of these cases is physically irrelevant and why?

(4.2) **Geodesics in Schwarzschild–Rindler**
Take the line-element

$$ds^2 = -K^2dt^2 + \frac{dr^2}{K^2} + r^2\,d\theta^2 + r^2\sin^2\theta\,d\phi^2\,, \qquad K^2 = 1 - \frac{2M}{r} + 2ar\,,$$

with Schwarzschild mass M and Rindler acceleration $-a$. Derive the effective potential V^{eff} that appears in the equation of time-like geodesics

$$\frac{\dot{r}^2}{2} + V^{\text{eff}} = E$$

with some $E = \text{const.}$ For the special case $|aM| \ll 1$ discuss perturbatively in aM how the radius of the innermost stable circular orbit (ISCO) is shifted as compared to Schwarzschild due to the Rindler term. For $aM \approx 10^{-23}$ how much is the radius of the ISCO shifted (in SI or Planck units) of a solar-mass black hole?

(4.3) **Christoffel symbols for Schwarzschild metric**
Take the Schwarzschild metric

$$g_{\mu\nu}dx^\mu dx^\nu = -(1-2M/r)\,dt^2 + \frac{dr^2}{1-2M/r} + r^2\,d\theta^2 + r^2\sin^2\theta\,d\phi^2$$

and calculate all Christoffel symbols of the second kind

$$\Gamma^\alpha{}_{\beta\gamma} = \frac{1}{2}\,g^{\alpha\mu}\big(\partial_\beta g_{\mu\gamma} + \partial_\gamma g_{\mu\beta} - \partial_\mu g_{\beta\gamma}\big)$$

These exercises are due on November 26^{th} 2013.

Hints:

- You may assume $\dot{\theta} = 0$, analog to the lectures. Then there are two routes: 1. Calculate the required Christoffel symbols of the second kind (see exercise 4.3 for a definition) and insert them into the geodesic equation we derived in the previous lectures. Note that you do *not* have to calculate all Christoffel symbols, but only a relatively small subset of them! 2. A convenient short-cut is to proceed exactly as we did in the lectures, except that you do not insert the Schwarzschild metric but its Newtonian approximation into the geodesic action.

- Use the same techniques as we did in the derivation of the Schwarzschild geodesics (e.g. $\dot{\theta} = 0$ and $\theta = \pi/2$). In particular, establish the existence of two constants of motion, like ℓ appearing in $\dot{\phi} = \ell/r^2$, and the fact that $g_{\mu\nu}\dot{x}^\mu x^\nu$ can be normalized to -1 for time-like geodesics. Remember that the definition of the ISCO requires the first two radial derivatives of the effective potential to vanish. Finally, Taylor expand in powers of aM and discard terms of $\mathcal{O}(aM)^2$ or smaller. You may cross-check with `arXiv:1103.0274`.

- This exercise is not difficult, since you just have to differentiate and sum various terms. However, it is quite some work. In fact, probably you will be bored at some point. But you really should do this calculation once in your lifetime. It helps you to appreciate the usefulness of Computer Algebra Systems and of more powerful formalisms in differential geometry.