Daniel Grumiller November 10^{th} 2015

Black Holes I — Exercise sheet 6

(6.1) **Riemann-tensor calculation**
Take the line element

$$ds^2 = -dt^2 + dr^2 + r^2\, d\theta^2 + r^2 \sin^2\theta\, d\phi^2$$

and calculate all Riemann-tensor components.

(6.2) **Abelian gauge field in curved and flat spacetime**
Given a dual (gauge) vector field A_μ and the covariant derivative ∇_μ we can construct various 2-tensors. Consider the antisymmetric tensor

$$F_{\mu\nu} = \nabla_\mu A_\nu - \nabla_\nu A_\mu$$

and show the identity

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \tag{1}$$

Assume next that we are in flat space. Vary the action $S = \int \mathrm{d}^4 x F_{\mu\nu} F^{\mu\nu}$ with respect to the gauge field A_μ, where the field strength $F_{\mu\nu}$ is related to the latter by Eq. (1) above. The equations you get are called "inhomogeneous Maxwell equations". The homogeneous Maxwell equations can be written as the vanishing of the 3-form $\mathrm{d}F = \epsilon^{\mu\nu\lambda\kappa}\partial_\mu F_{\nu\lambda} = 0$, where $\epsilon^{\mu\nu\lambda\kappa}$ is the totally antisymmetric ε-tensor. Finally, split the field strength into two spatial "vectors" as follows: $E_i := F_{i0}$, $B_k := \frac{1}{2}\,\epsilon_{ijk} F_{ij}$ and write all equations in terms of E_i and B_i. Does this look familiar?

(6.3) **Lie derivatives**
Given the three vector fields

$$\begin{aligned}
L_0 &= \partial_\phi \\
L_1 &= -\cos\phi\,\partial_\theta + \sin\phi\cot\theta\,\partial_\phi \\
L_2 &= \sin\phi\,\partial_\theta + \cos\phi\cot\theta\,\partial_\phi
\end{aligned}$$

calculate their Lie-brackets (commutators)

$$[L_i, L_j] = f_{ij}{}^k\, L_k$$

and determine the structure constants $f_{ij}{}^k$. (If you know about Lie algebras try to find out which Lie algebra is generated by these three vector fields). Calculate also the Lie derivatives

$$\mathcal{L}_{L_i}\left(g_{\alpha\beta}\right) = L_i^\mu \partial_\mu g_{\alpha\beta} + g_{\alpha\mu}\partial_\beta L_i^\mu + g_{\mu\beta}\partial_\alpha L_i^\mu$$

of the metric of the unit sphere,

$$ds^2 = g_{\alpha\beta}\, dx^\alpha dx^\beta = d\theta^2 + \sin^2\theta\, d\phi^2$$

along all three vector fields L_i given above. What is the meaning of your result?

These exercises are due on November 24^{th} 2015

Hints:

- Either you calculate all Christoffels and insert into the definition of the Riemann-tensor (easy, but lengthy), or you make a suitable coordinate transformation to a simpler set of coordinates, calculate the Riemann-tensor $R^\alpha{}_{\beta\gamma\delta}$ in these simpler coordinates and use the fact that the Riemann-tensor transforms as a $(1,3)$ tensor.

- For the first part just insert the definition of the covariant derivative acting on a dual vector and prove that the connection terms cancel. For the second part remember that we are here in flat spacetime, so all Greek indices are raised and lowered with the Minkowski metric and Latin indices with the Euclidean metric (thus we do not have to raise Latin indices at all). Your result for the inhomogeneous Maxwell equations should read $\partial_\mu F^{\mu\nu} = 0$ (they are called "inhomogeneous", because the right hand side may contain a source term j^ν once we add interactions with charged matter). For the final part, if you want to bring this into a form that is probably familiar to you remember the definitions $\partial_i E_i = \operatorname{div} E$ and $\epsilon_{ijk}\partial_j E_k = (\operatorname{curl} E)_i$. Also, do not hesitate to multiply equations by convenient quantities, for instance ϵ_{ijk}. Occassionally you might want to use the identity $\epsilon_{ijk}\epsilon_{inm} = \delta_{jn}\delta_{km} - \delta_{jm}\delta_{kn}$.

- The definition of a commutator between two vector fields is $[A, B] = AB - BA$. A simple example for illumination: Take the vector fields $\xi = x\,\partial_x + y\,\partial_y$ and $\zeta = c\,\partial_x$, with $c = \text{const}$. Then their Lie bracket is $[\xi, \zeta] = x(\partial_x c)\partial_x + y(\partial_y c)\partial_x - c((\partial_x x)\partial_x + (\partial_x y)\partial_y) = -c\,\partial_x = -\zeta$. Regarding the final question of this exercise: remember what we learned in the lectures about symmetries and Killing vectors.