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Black Holes II — Exercise sheet 4

(14.1) **High-frequency limit of Regge–Wheeler equation**
The Regge–Wheeler equation can be approximated by a confluent hypergeometric equation in the high-frequency limit $\omega M \to \infty$. Solving this equation one can show that the asymptotic amplitudes appearing in the solution $\hat{u}_\ell^{\text{in}}(r_*, \omega)$ are given by

$$A_{\text{out}} \approx \frac{\Gamma(1-4i\omega M)}{\sqrt{2\pi}(4i\omega M)^{1/2-4i\omega M}}\, e^{-4i\omega M} \qquad A_{\text{in}} \approx \frac{i\Gamma(1-4i\omega M)}{\Gamma(1/2-4i\omega M)\sqrt{4i\omega M}}$$

Use Stirling's formula to show that the reflection coefficient in the high-frequency limit $\omega M \to \infty$ behaves as

$$|R|^2 \approx e^{-8\pi\omega M}$$

Interpret this result concisely.

(14.2) **Second order field equations for 2D dilaton gravity**
Derive the equations of motion by varying the 2D dilaton gravity action

$$S^{\text{2DG}} = \frac{1}{\kappa}\int \mathrm{d}^2x\sqrt{-g}\,\Big[XR - U(X)(\nabla X)^2 - 2V(X)\Big]$$

with respect to the dilaton field X and the metric $g_{\mu\nu}$ (U, V are arbitrary functions of X). You may neglect surface terms in this exercise. Compare your result with Eqs. (2.1) and (2.2) in `hep-th/0703230`.

(14.3) **Asymptotic quasi-normal modes in 2D dilaton gravity**
Take the dilaton gravity action from exercise (14.2) and couple it non-minimally to a massless scalar field ϕ

$$S = S^{\text{2DG}} + S^{\text{mat}} \qquad S^{\text{mat}} = -\frac{1}{2}\int \mathrm{d}^2x\sqrt{-g}\,X^p\,(\nabla\phi)^2$$

with constant p. Using the monodromy approach by Motl and Neitzke (`gr-qc/0212096` and `hep-th/0301173`), the quasi-normal mode spectrum of this model was analyzed in `gr-qc/0408042` in the limit of large damping. For the complex frequency ω the asymptotic relation

$$e^{\omega/T_H} = -\big(1 + 2\cos\left(\pi(1-p)\right)\big)$$

was found, where T_H is the Hawking temperature of the black hole. Asymptotic means that the imaginary part of ω/T_H is large and positive. Consider the minimally coupled case ($p = 0$) and the Schwarzschild case ($p = 1$) and derive formulas for the real and imaginary parts of ω/T_H. Compare the $p = 1$ case with the (computer-) experimental results for the Schwarzschild black hole by Nollert (*Phys. Rev.* **D47** (1993) 5253) and Andersson (*Class. Quant. Grav.* **L10** (1993) 61) who found the asymptotic formula

$$\omega M \approx 0.0437 + \frac{i}{4}\,(n + \frac{1}{2})$$

Does the 2D dilaton gravity formulation of the Schwarzschild black hole lead to the correct asymptotic spectrum for quasi-normal modes?

These exercises are due on April 19th 2010.
NOTE: NO LECTURES ON APRIL 12th!

Hints:

- Remember our definition of the in-mode, $\hat{u}_\ell^{\rm in}(r_*, \omega) \sim e^{-i\omega r_*}$ for $r_* \to -\infty$ and

$$\hat{u}_\ell^{\rm in}(r_*, \omega) \sim A_{\rm out}(\omega)\, e^{i\omega r_*} + A_{\rm in}(\omega)\, e^{-i\omega r_*} \qquad \text{for } r_* \to \infty$$

  Remember also the standard definition of the reflection coefficient

$$R = \frac{A_{\rm out}}{A_{\rm in}}$$

  For your convenience, the Stirling formula for complex z can be presented as

$$\ln\Gamma(z) = (z - \frac{1}{2})\,\ln z - z + \frac{\ln 2\pi}{2} + \mathcal{O}(1/z)$$

  provided the real part of z is positive. Note that $\ln(-1 - i\varepsilon) = -i\pi + i\varepsilon + \mathcal{O}(\varepsilon^2)$ for small but positive ε.

- The dilaton variation is straightforward [analog to exercise (3.2c)]. For the metric variation use the formula $\delta\sqrt{-g} = \frac{1}{2}\sqrt{-g}\, g^{\mu\nu}\,\delta g_{\mu\nu}$ (see previous semester). As we showed last semester (see also later this semester) the variation of the Ricci scalar yields

$$\delta R = -R^{\mu\nu}\,\delta g_{\mu\nu} + \nabla^\mu\nabla^\nu\,\delta g_{\mu\nu} - g^{\mu\nu}\nabla^2\,\delta g_{\mu\nu}$$

  Exploit also the fact that the 2D Einstein tensor vanishes identically for any 2D metric, $R_{\mu\nu} = \frac{1}{2}\, g_{\mu\nu} R$. Be careful with signs!

- This is an exceedingly short exercise. You only have to know that $T_H = 1/(8\pi M)$ for the Schwarzschild black hole. (We motivated this result last semester, see exercise (9.3), and we shall derive it from scratch this semester.)