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Black Holes II — Exercise sheet 6

(16.1) **Higher curvature theories of gravity**
Show that a non-linear gravity theory in any dimension $D \geq 2$ with an action

$$\tilde{S} = \frac{1}{\tilde{\kappa}} \int \mathrm{d}^D x \sqrt{-g}\, R^{\tilde{\gamma}} \qquad \tilde{\gamma} \neq 0, 1$$

is equivalent to the following class of dilaton gravity models:

$$S = \frac{1}{\kappa} \int \mathrm{d}^D x \sqrt{-g} \Big[XR - X^{\gamma} \Big]$$

Derive a relation between the exponents γ and $\tilde{\gamma}$.
[Note: This result shows the equivalence of theories with $f(R)$ interactions to certain dilaton gravity models. The latter are also known as scalar-tensor theories, Jordan–Brans–Dicke theories or quintessence models. Both models have been used a lot in cosmology in the past 1.5 decades.]

(16.2) **Field equations for spherically symmetric charged BHs**
Requiring spherical symmetry at the level of the action reduces the Einstein–Hilbert–Maxwell action in 4D to a specific dilaton gravity-Maxwell action in 2D (to reduce clutter we set $\kappa = 1$):

$$S^{\text{RNBH}} = \int \mathrm{d}^2 x \sqrt{-g} \Big[XR + \frac{(\nabla X)^2}{2X} + 1 \Big] - \int \mathrm{d}^2 x \sqrt{-g}\, X F_{\mu\nu} F^{\mu\nu}$$

Derive the equations of motion for metric g, dilaton X and gauge-field A_μ (as usual $F_{\mu\nu} = \nabla_\mu A_\nu - \nabla_\nu A_\mu = \partial_\mu A_\nu - \partial_\nu A_\mu$). Solve for $F_{\mu\nu}$ using the field equations for A_μ. Interpret this result from a 4D perspective. What is the physical role of the constant of motion appearing in the solution of the field equations for A_μ?

(16.3) **Deriving the Reissner-Nordström solution**
Take the equations of motion derived in (16.2) and find their most general solution for the metric g, non-constant dilaton X and gauge-field A_μ. Oxidize your result to 4D and write down the line-element in Schwarzschild coordinates. This line-element is the Reissner-Nordström solution. Finally, find all constant dilaton vacua. Are these vacuum geometries $\text{AdS}_2 \times S^2$, Minkowski $\times S^2$ or $\text{dS}_2 \times S^2$?

These exercises are due on May 3rd 2010.

Hints:

- Start with the dilaton gravity formulation and eliminate the dilaton X in terms of curvature R by means of its own equation of motion.

- The field equations for dilaton and metric can be derived essentially in the same way as for exercise (14.2) — see the hints there. For consistency your field equations in the limit $A_\mu \to 0$ must coincide with the ones derived in (14.2) for the special case $U(X) = -\frac{1}{2X}$ and $V(X) = -\frac{1}{2}$. The field equation for the gauge field is straightforward. You should obtain
$$\nabla_\mu \left(X\, F^{\mu\nu}\right) = \varepsilon^{\mu\nu}\, \partial_\mu \left(X\, f\right) = 0$$
where the first equality exploits the fact that any anti-symmetric tensor $F^{\mu\nu}$ in 2D can be written as $F^{\mu\nu} = \varepsilon^{\mu\nu}\, f$, where f is a scalar field and $\varepsilon^{\mu\nu}$ the ε-tensor. It is straightforward to solve the field equations above for f in terms of X and an integration constant. Regarding the 4D interpretation remember how the dilaton field X is related to the standard radial coordinate r and compare with the Coulomb solution.

- If you did not do exercise (16.2) then you need to know the field equations. The one for A_μ is provided in the hint above. The one for the dilaton reads
$$R + \frac{(\nabla X)^2}{2X^2} - \frac{\nabla^2 X}{X} - F^2 = 0$$
where $F^2 = F_{\mu\nu}F^{\mu\nu}$. Variation with respect to the metric yields
$$\nabla_\mu \nabla_\nu X - g_{\mu\nu}\, \nabla^2 X - \frac{(\nabla_\mu X)(\nabla_\nu X)}{2X} + g_{\mu\nu}\, \frac{(\nabla X)^2}{4X} + \frac{1}{2}\, g_{\mu\nu} = -2X\, T_{\mu\nu}$$
with
$$T_{\mu\nu} = F_{\mu\alpha} F_\nu{}^\alpha - \frac{1}{4}\, g_{\mu\nu}\, F^2 = -\frac{1}{2}\, g_{\mu\nu}\, f^2$$
In the last equality I have used $F_{\mu\nu} = \varepsilon_{\mu\nu}\, f$ as well as the 2D identity $\varepsilon_{\mu\alpha}\varepsilon^\alpha{}_\nu = g_{\mu\nu}$ (for Minkowski signature). Exploit now the fact that you can introduce an effective 2D dilaton gravity model with potentials
$$U^{\text{eff}} = U = -\frac{1}{2X} \qquad V^{\text{eff}} = -\frac{1}{2} + \frac{q^2}{4X}$$
where q is the constant of motion appearing in the dual field strength $f = q/(2X)$. This trick allows you to take advantage of the results derived in the lectures for general 2D dilaton gravity solutions with non-constant dilaton [see also the hint for (15.1)]. For the oxidation remember that the 4D line element is determined from the 2D metric $g_{\alpha\beta}$ and the dilaton X as follows:
$$\mathrm{d}s^2_{(4)} = g_{\alpha\beta}\, \mathrm{d}x^\alpha\, \mathrm{d}x^\beta + 2X\, \mathrm{d}\Omega^2_{S^2}$$
For the constant dilaton vacua use $X = \text{const.}$ as early as possible. Exploit that the Ricci scalar R uniquely determines the Riemann tensor in 2D — if you know e.g. that R is constant spacetime can only be deSitter, Minkowski or Anti-deSitter, depending on the sign of R.