Daniel Grumiller May 10th 2010

Black Holes II — Exercise sheet 8

(18.1) **Schwarzschild black hole and thermal reservoir**
Couple the Schwarzschild black hole to a finite thermal reservoir of radiation within the volume V at the Hawking temperature $T_H = 1/(8\pi M)$. Show that for sufficiently small volumina $V < V_c$ this system is thermodynamically stable, whereas for sufficiently large volumina $V > V_c$ it is unstable. Calculate V_c.

(18.2) **Specific heat of Reissner–Nordström black hole**
Calculate the Bekenstein–Hawking entropy S_{BH} for the Reissner–Nordström black hole and the specific heat C at fixed charge q,

$$C = T_H \frac{\partial S_{BH}}{\partial T_H}\Big|_{q=\text{const.}}$$

where T_H is the Hawking temperature. Discuss the thermodynamic stability of the Reissner–Nordström black hole for

(i) $q^2 \leq M^2 < \frac{4}{3}q^2$
(ii) $M^2 > \frac{4}{3}q^2$

(18.3) **Information loss problem in condensed matter physics**
Consider a piece of coal at zero temperature and a laser beam (a pure quantum state with some finite energy and entropy) in vacuum as initial state. Provided the laser beam is directed toward the piece of coal it will eventually be absorbed and scattered by the coal. In this (complicated) process the coal will heat up a little bit. Suppose that the coal is a nearly perfect black body. Then the final state will be the scattered pure radiation and the outgoing thermal black-body radiation emitted by the piece of coal. Thus, we appear to have an evolution of a pure initial state into a final state that is not pure. Information is lost, similar to what happens in the case of an evaporating black hole. How is this information loss problem resolved in condensed matter physics?

These exercises are due on May 17th 2010.

Hints:

- For the finite reservoir of radiation you need the Stefan–Boltzmann law $E_{\rm res} = \sigma V T^4$, where $E_{\rm res}$ is the energy of the radiation and $\sigma = \pi^2/15$. The relation between energy $E_{\rm res}$ and entropy $S_{\rm res}$ for a radiation gas is given by $E_{\rm res} = \frac{3}{4} S_{\rm res} T$. Use the Bekenstein–Hawking result for the entropy, $S_{BH} = A/4$, and show that the total entropy $S = S_{\rm res} + S_{BH}$ is extremized for a total energy of $E = E_{\rm res} + M$ if $T = T_H$. A simple way to extremize entropy under the given conditions is to add to the total entropy the energy constraint multiplied with a Lagrange multiplier β. Then vary that entropy with respect to the Lagrange multiplier and with respect to the black hole mass, keeping fixed the total energy E:

  $$\delta S = \delta\big(S_{\rm res} + S_{BH} + \beta(E_{\rm res} + M - E)\big) = 0$$

  Prove now that the extremum is a maximum if and only if $V < V_c$, where

  $$V_c = \frac{15}{32\pi^3 T^5}$$

  Consider what this result implies for thermodynamic (in-)stability.

- Recall that a thermodynamical system in the canonical ensemble is unstable if the specific heat is negative. Calculate the critical value of the mass in terms of charge when the specific heat changes its sign. This can be a very short exercise, but if you take a less convenient way to derive it it can also be a bit lengthy. Technically, a simple way to derive the desired result is by means of the 2D dilaton gravity formulation [see exercise (16.2)]. Use the results for $U(X), V(X)$ given in the hints of exercise (16.3) and exploit the result that we derived for the specific heat in 2D dilaton gravity:

  $$C = 2\pi \frac{w'(X)}{w''(X)}\Big|_{X=r_+^2/2}$$

  where $r_+ = M + \sqrt{M^2 - q^2}$ is the locus of the event horizon in Schwarzschild coordinates and $w(X)$ is related to the potentials $U(X), V(X)$ as defined in the hint of exercise (15.1).

- Think. Perhaps compare with exercise (8.3). Think again.