Daniel Grumiller May 17th 2010

Black Holes II — Exercise sheet 9

(19.1) **Hamilton–Jacobi formulation**
Recall the Hamilton–Jacobi formulation of mechanics by deriving the Hamilton–Jacobi equation for Hamilton's principal function $S(q,t)$ for a given Hamiltonian $H(q,p)$

$$H\big(q, \frac{\partial S}{\partial q}\big) + \frac{\partial S}{\partial t} = 0$$

You can start either with the Hamilton formulation or the Lagrange formulation or the Newton formulation of mechanics.

(19.2) **Holographic renormalization in quantum mechanics**
Take the Hamiltonian of conformal quantum mechanics [V. de Alfaro, S. Fubini and G. Furlan, *Nuovo Cim.* **A34** (1976) 569]

$$H(q,p) = \frac{p^2}{2} + \frac{1}{q^2}$$

for $q > 0$. Consider the variational principle for the action

$$I[q] = \int_{t_0}^{t_1} \mathrm{d}t \, \big(\frac{\dot{q}^2}{2} - \frac{1}{q^2}\big)$$

in the limit $t_1 \to \infty$ and holographically renormalize the action. Discuss what happens with the first variation of the action if you do *not* add a holographic counterterm.

(19.3) **On-shell action of 2D dilaton gravity**
Calculate the on-shell action for Euclidean 2D dilaton gravity with an asymptotic boundary at $X = \infty$, where X is the dilaton field.

These exercises are due on May 31st 2010.

Hints:

- Check any book on theoretical mechanics if you need a reminder. It is sufficient for this exercise to derive the Hamilton–Jacobi equation for the simplest case possible.

- Make the Ansatz for the improved action

$$\Gamma = \int\limits_{t_0}^{t_1} \mathrm{d}t \,\Big(\frac{\dot{q}^2}{2} - \frac{1}{q^2}\Big) - S(q,t)\Big|_{t_0}^{t_1}$$

  and postulate that the counterterm solves the Hamilton–Jacobi equation

$$H\big(q, \frac{\partial S}{\partial q}\big) + \frac{\partial S}{\partial t} = 0$$

  Solve the Hamilton–Jacobi equation asymptotically (in the limit of large t_1) and keep the first term (if you want you may keep also the subleading term). If you need further hints consult `0711.4115` where exactly this example is treated.

- The full action for Euclidean 2D dilaton gravity is given by

$$\Gamma = -\frac{1}{16\pi G}\int_{\mathcal{M}} d^2x\,\sqrt{g}\,\big[X\,R - U(X)\,(\nabla X)^2 - 2\,V(X)\big] - \frac{1}{8\pi G}\int_{\partial\mathcal{M}} dx\,\sqrt{\gamma}\,X\,K + \frac{1}{8\pi G}\int_{\partial\mathcal{M}} dx\,\sqrt{\gamma}\,\sqrt{w(X)\,e^{-Q(X)}}\,,$$

  with the standard definitions of the functions w and Q in terms of the potentials U and V [see hint of exercise (15.1)]. Recall that the solutions of the equations of motion are given by

$$X = X(r) \qquad \mathrm{d}s^2 = \xi(r)\,\mathrm{d}\tau^2 + \frac{1}{\xi(r)}\,\mathrm{d}r^2$$

  with

$$\partial_r X = e^{-Q(X)} \qquad \xi(X) = w(X) e^{Q(X)}\,\big(1 - \frac{2M}{w(X)}\big)$$

  where M is the black hole mass. Since the boundary is an $X = \text{const.}$ hypersurface, in the gauge above the determinant of the induced metric at the boundary is given by $\gamma = \xi(X)$, while the trace of extrinsic curvature is given by

$$K = \frac{1}{2\sqrt{\xi}}\,\frac{\partial \xi}{\partial r}$$

  Note that you are allowed to exploit on-shell identities to simplify the calculation.