Daniel Grumiller May 31st 2010

Black Holes II — Exercise sheet 10

(20.1) **Thermodynamics of the ab-family**
Take the on-shell action for the ab-family in Euclidean 2D dilaton gravity and discuss thermodynamical properties depending on the parameters a and b. In particular, discuss for which values of a and b specific heat is always positive regardless of the cut-off. You can compare your results with section 4.2 in `hep-th/0703230`.

(20.2) **Hawking–Page phase transition**
Consider spherically reduced AdS$_D$-gravity, i.e., 2D dilaton gravity with potentials

$$U = -\frac{D-3}{(D-2)X} \qquad V = -\frac{(D-1)(D-2)}{2\ell^2}\,X - A\,X^{(D-4)/(D-2)}$$

with some positive constants ℓ, A and $D > 3$. Show that the specific heat of black hole solutions in AdS$_D$ is positive for sufficiently large black holes, but negative for small black holes. Quantify precisely what "large" and "small" means in this context.

(20.3) **Asymptotic symmetry group in AdS$_3$ quantum gravity**
Given some boundary conditions for the metric, the asymptotic symmetry group consists of those diffeomorphisms that preserve these boundary conditions (modulo trivial diffeomorphisms). Consider the Brown–Henneaux boundary conditions [J. D. Brown and M. Henneaux, *Commun. Math. Phys.* **104** (1986) 207] near $y = 0$,

$$g_{\alpha\beta} = \begin{pmatrix} g_{++} = \mathcal{O}(1) & \frac{\ell^2}{2y^2} + \mathcal{O}(1) & \mathcal{O}(y) \\ g_{-+} = g_{+-} & g_{--} = \mathcal{O}(1) & \mathcal{O}(y) \\ g_{y+} = g_{+y} & g_{y-} = g_{-y} & g_{yy} = \frac{\ell^2}{y^2} + \mathcal{O}(1) \end{pmatrix}$$

where the total metric $g = \bar{g} + h$ consists of an asymptotic AdS$_3$ background

$$\bar{g}_{\alpha\beta}\,\mathrm{d}x^\alpha\,\mathrm{d}x^\beta = \ell^2\,\frac{\mathrm{d}x^+\,\mathrm{d}x^- + \mathrm{d}y^2}{y^2}$$

and of fluctuations h that fall off near $y = 0$ according to the above boundary conditions. Show that the asymptotic symmetry group is given by diffeomorphisms generated by a vector field ξ of the form

$$\xi^+ = \varepsilon^+(x^+) - \frac{y^2}{2}\,\partial_-^2\varepsilon^- + \mathcal{O}(y^4)$$
$$\xi^- = \varepsilon^-(x^-) - \frac{y^2}{2}\,\partial_+^2\varepsilon^+ + \mathcal{O}(y^4)$$
$$\xi^y = \frac{y}{2}\left(\partial_+\varepsilon^+(x^+) + \partial_-\varepsilon^-(x^-)\right) + \mathcal{O}(y^3)$$

where $\varepsilon^\pm$ are arbitrary functions of their arguments.

These exercises are due on June 7th 2010.

Hints:

- Recall the definition of the *ab*-family given in exercise (15.1) [see also the corresponding hint] and the result for the on-shell action Γ_c evaluated at a certain cut-off for the dilaton X_c, which we derived in the lectures and in exercise (19.3):

$$\Gamma_c = \beta \left(w(X_c) \sqrt{1 - \frac{2M}{w(X_c)}} - w(X_c) + 2\,M - 2\,\pi X_H\,T \right)$$

  Here $\beta = T^{-1} = 4\pi/w'(X_H)$ is the inverse periodicity in Euclidean time and X_H is the value of the dilaton at the horizon, $w(X_H) = 2M$. Note that the Helmholtz free energy is related to the on-shell action by

$$F_c = T_c\,\Gamma_c$$

  where $T_c = T/\sqrt{\xi_c}$ is the red-shifted temperature, with $\xi_c = e^{Q_c}\,w_c(1 - 2M/w_c)$. Entropy S is given by the derivative of free energy F_c with respect to temperature T_c, keeping fixed the cut-off X_c. You should recover the result that entropy is independent from the cut-off X_c. Specific heat is given by $T_c \partial S/\partial T_c$, keeping fixed the cut-off X_c. You should recover in this way the result (3.28) of `hep-th/0703230`.

- Use the hints of exercise (18.2), except that you should replace "charge" by "AdS-radius ℓ" in the text and that you have to calculate the locus of the black hole horizon r_+ from scratch. Exploit in particular the formula for the specific heat provided in that hint, and consult exercise (15.1) and the hints therein if you need a reminder of the 2D dilaton gravity formulation.

- Recall that a diffeomorphism generated by a vector field ξ acts on the metric via the Lie derivative

$$\mathcal{L}_\xi\, g_{\mu\nu} = \xi^\sigma \partial_\sigma g_{\mu\nu} + g_{\mu\sigma}\partial_\nu \xi^\sigma + g_{\nu\sigma}\partial_\mu \xi^\sigma$$

  Check now that the diffeomorphisms generated by the vector field ξ given at the end of (20.3) preserve the Brown–Henneaux boundary conditions. As an example here is how you check that the $++$-component is ok:

$$\mathcal{L}_\xi g_{++} = \xi^\mu \partial_\mu g_{++} + 2 g_{+\mu}\partial_+ \xi^\mu = 2 g_{+-}\partial_+ \xi^- + \mathcal{O}(1) = -\frac{\ell^2}{y^2}\frac{y^2}{2}\,\partial_+^2 \varepsilon^+ + \mathcal{O}(1) = \mathcal{O}(1)$$