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Black Holes II — Exercise sheet 3

(13.1) **Schwarzschild black holes do not bifurcate**
We saw last semester that a black hole binary system can lead to a single black hole final state, after emitting an appreciable amount of gravitational radiation in the inspiralling, merger and ringdown phases. The endstate was a Kerr black hole with mass M and angular momentum J. We also saw that a Kerr black hole is unstable against superradiance, which lowers its angular momentum and mass, while maintaining (or increasing) its horizon area. The endstate of this chain of instabilities is a Schwarzschild black hole. Prove that this endstate is classically stable against bifurcation. In other words, show that a Schwarzschild black hole cannot decay into two (Schwarzschild or Kerr) black holes.[1]

(13.2) **Raychaudhuri equation for geodesic null congruences**
Consider a congruence of null geodesics with tangent vector field k^a and deviation vector η^a (in 4 spacetime dimensions). Derive the analog of the Raychaudhuri equation for the expansion θ. You should find

$$k^a \nabla_a \theta = -\frac{1}{2}\,\theta^2 - \sigma_{ab}\sigma^{ab} + \omega_{ab}\omega^{ab} - R_{ab}k^a k^b$$

Explain why there is a factor $\frac{1}{2}$ instead of $\frac{1}{3}$.

(13.3) **Prove Your Own Singularity Lemma**
Given a geodesic null congruence with vanishing twist ($\omega_{ab} = 0$) and assuming the null energy condition (so that $R_{ab}k^a k^b \geq 0$) show the following lemma: If the expansion is negative, $\theta = \theta_0 < 0$, at some point on a geodesic in the congruence then $\theta \to -\infty$ along that geodesic within the affine distance $\lambda \leq 2/|\theta_0|$.

These exercises are due on March 29th 2012.

[1] Unless, of course, something provides sufficient energy to trigger such a process. But when we use the word “decay” we usually mean “decay all by itself, without some external trigger”. Sidenote: the black holes that we observe have a surrounding accretion disk, which actually tends to bring the rotating black hole towards extremality; the processes described in this exercise work in the other direction and apply to black holes in vacuum.

Hints:

- Use Hawking's area theorem. Use energy conservation. Perhaps have a look at exercises (10.2) and (10.3). This is a short exercise!

- Note that the orthogonality condition $k^a\eta_a = 0$ is not sufficient to determine uniquely η^a since the displacement vector may now have a component parallel to k^a. Thus, the displacement vector η^a orthogonal to the tangent vector k^a specifies only a co-dimension 2 parameter family of geodesics (in our 4 spacetime dimensions this is a 2-parameter family). Therefore, you must specify one further condition to fix the displacement vector η^a. It is convenient to introduce another null-vector l^a with the properties $l^a l_a = 0$, $l^a k_a = -1$ and $k^a \nabla_a l^b = 0$ (so if k^a is an ingoing null vector l^a is an outgoing one, with some convenient normalization). Then you may impose the condition

  $$\eta^a l_a = 0$$

  Instead of the "spatial metric" $h_{ab} = g_{ab} + t_a t_b$ introduce now the projector

  $$P_{ab} = g_{ab} + k_a l_b + l_a k_b$$

  that projects onto the required co-dimension 2 subspace of the tangent space, $P^a_b \eta^b = \eta^a$. The tensor field $B_{ab} = \nabla_b k_a$ must also be projected,

  $$\hat{B}^a{}_b := P^a_c B^c{}_d P^d_b$$

  After showing that $k^a\nabla_a\eta^b = \hat{B}^b{}_a\eta^a$ (remember that $k^a\nabla_a\eta^b = \eta^a\nabla_a k^b$) make the decomposition

  $$\hat{B}^a{}_b = \frac{1}{2}\,\theta P^a_b + \sigma^a{}_b + \omega^a{}_b$$

  and think about the correctness of the factor $\frac{1}{2}$. The rest is completely analog to the derivation of the Raychaudhuri equations for timelike geodesic congruences. This is a long exercise!

- This lemma is the light-like analog of the lemma we proved for time-like geodesic congruences during the lectures. You need the result for the Raychaudhuri equation for geodesic null congruences spelled out in exercise (13.2), and otherwise proceed analog to the lectures.