Daniel Grumiller March 29th 2012

Black Holes II — Exercise sheet 4

(14.1) **Hamilton–Jacobi formulation**
Recall the Hamilton–Jacobi formulation of mechanics by deriving the Hamilton–Jacobi equation for Hamilton's principal function $S(q,t)$ for a given Hamiltonian $H(q,p)$

$$H\big(q, \frac{\partial S}{\partial q}\big) + \frac{\partial S}{\partial t} = 0$$

You can start either with the Hamilton formulation or the Lagrange formulation or the Newton formulation of mechanics.

(14.2) **Holographic renormalization in quantum mechanics**
Take the Hamiltonian of conformal quantum mechanics [V. de Alfaro, S. Fubini and G. Furlan, *Nuovo Cim.* **A34** (1976) 569]

$$H(q,p) = \frac{p^2}{2} + \frac{1}{q^2}$$

for $q > 0$. Consider the variational principle for the action

$$I[q] = \int\limits_{t_0}^{t_1} \mathrm{d}t\,\big(\frac{\dot{q}^2}{2} - \frac{1}{q^2}\big)$$

in the limit $t_1 \to \infty$ and holographically renormalize the action.

(14.3) **Consistency of classical approximation**
Classical field theories (depending on some fields collectively denoted by ϕ) arise from quantum field theories in the formal limit $\hbar \to 0$. Discuss what goes wrong with this limit if you consider a holographically *unrenormalized* action $\Gamma[\phi]$, whose first variation is not zero if evaluated on solutions of the classical equations of motion (EOM)

$$\delta\Gamma[\phi]\big|_{\text{EOM}} \neq 0$$

for some variations that preserve the boundary conditions of the theory.

These exercises are due on April 26th 2012.

Hints:

- Check any book on theoretical mechanics if you need a reminder. It is sufficient for this exercise to derive the Hamilton–Jacobi equation for the simplest case possible.

- Make the Ansatz for the improved action

$$\Gamma[q] = \int_{t_0}^{t_1} \mathrm{d}t \, \Big(\frac{\dot{q}^2}{2} - \frac{1}{q^2}\Big) - S(q,t)\Big|_{t_0}^{t_1}$$

  and postulate that the counterterm solves the Hamilton–Jacobi equation

$$H\Big(q, \frac{\partial S}{\partial q}\Big) + \frac{\partial S}{\partial t} = 0$$

  Solve the Hamilton–Jacobi equation asymptotically (in the limit of large t_1) and keep the first term (if you want you may keep also the subleading term). If you need further hints consult `0711.4115` where exactly this example is treated.

- The action evaluated on some generic field configuration $\phi = \phi_c + \delta\phi$ can be expanded around ϕ_c, which is some solution to the classical equations of motion:

$$\Gamma[\phi_c + \delta\phi] = \Gamma(\phi_c) + \frac{\delta\Gamma}{\delta\phi}\Big|_{\phi=\phi_c} \delta\phi + \mathcal{O}(\delta\phi^2)$$

  In the formal limit $\hbar \to 0$ the terms of $\mathcal{O}(\delta\phi^2)$ can be neglected.[1] However, for the consistency of the classical approximation it is imperative that the term linear in $\delta\phi$ vanishes identically, for all possible $\delta\phi$ that are compatible with the boundary conditions imposed on ϕ. You should discuss *why* it is necessary for the linear term to vanish (this might be easiest in the path integral language) and also explain how it is possible for this term to be non-vanishing, even though ϕ_c by definition is a solution to the classical equations of motion.

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[1]This statement is true only if the sign of the quadratic fluctuations is non-negative, so there can be (and indeed are) further subtle issues, particularly in the context of black hole thermodynamics. However, for the purpose of this exercise you are allowed to neglect these issues and focus on the leading two terms in the expansion of the action.