Daniel Grumiller March 29th 2012

Black Holes II — Exercise sheet 5

(15.1) **Extrinsic curvature of cylinder**
Take flat Euklidean space, roll it into a cylinder and calculate the extrinsic curvature tensor and its trace at the boundary.

(15.2) **Gauss–Bonnet theorem with an edge**
Consider the round 2-sphere and calculate the expression on the left hand side of the simplest version of the Gauss–Bonnet theorem

$$\frac{1}{4\pi}\int_{S^2} \mathrm{d}^2x\sqrt{g}\,R = \chi(S^2)$$

The right hand side is known as Euler characteristic of the corresponding manifold, $\chi(M) = 2(1-g)$, where g is the genus of the manifold M. What is the resulting g for the 2-sphere? Consider now only half of the 2-sphere and calculate the expression

$$\frac{1}{4\pi}\int_{M} \mathrm{d}^2x\sqrt{g}\,R + \frac{1}{2\pi}\int_{\partial M} \mathrm{d}x\sqrt{\gamma}\,K$$

where M is now half of the 2-sphere (including the disk), ∂M its boundary, γ the induced metric at the boundary and K trace of extrinsic curvature. How does this result compare with the previous one?

(15.3) **Generalized Fefferman–Graham expansion**
Consider an asymptotically AdS metric in Gaussian normal coordinates

$$\mathrm{d}s^2 = \mathrm{d}\rho^2 + \gamma_{ij}\,\mathrm{d}x^i\,\mathrm{d}x^j$$

with the following asymptotic expansion in the limit of large ρ

$$\gamma_{ij} = e^{2\rho}\gamma_{ij}^{(0)} + e^{\rho}\gamma_{ij}^{(1)} + \gamma_{ij}^{(2)} + \dots$$

where $\gamma_{ij}^{(0)} = \eta_{ij}$ is the flat Minkowski metric. Calculate the asymptotic expansion for extrinsic curvature K_{ij} and its trace $K = K_{ij}\gamma^{ij}$. It is sufficient to keep only the three leading terms in these expansions (note that some of the terms may vanish; in that case you need *not* go to even higher order in the expansion).

These exercises are due on April 26th 2012.

Hints:

- It is easiest if you use Gaussian normal coordinates to parametrize the cylinder, e.g. standard cylindrical coordinates

$$\mathrm{d}s^2 = \mathrm{d}\rho^2 + \rho^2\,\mathrm{d}\phi^2 + \mathrm{d}z^2$$

  The boundary is a $\rho = $ const. hypersurface so that $\rho = \rho_0$ at the boundary. Derive then the induced metric at the boundary γ_{ij} (where $x^i = \{\phi,\, z\}$) and exploit that in Gaussian normal coordinates extrinsic curvature is given by

$$K_{ij} = \frac{1}{2}\,\partial_\rho \gamma_{ij}$$

  Its trace is then given by $K = K_{ij}\gamma^{ij}$.

- The most important part of this exercise is to realize that the manifold consists of two separate pieces, namely the half of the sphere, say, the southern hemisphere, and the disk on top. You can picture this by mentally cutting through Earth and throwing away they northern hemisphere. As you should check, one part turns out to be intrinsically flat and extrinsically curved, while the other part is extrinsically flat and intrinsically curved. For the round unit sphere the Ricci scalar is given by $R = 2$, as you can check easily. (If you get $R = -2$ then you are using other conventions and have to use different signs in the Gauss–Bonnet formulas! I am using conventions such that $R_{\mu\nu} = +\partial_\lambda \Gamma^\lambda{}_{\mu\nu} - \dots$) The extrinsic curvature of the disk works essentially in the same way as in the first exercise.
  Historical sidenote: The Gauss–Bonnet theorem is the simplest example of an index theorem, a type of theorem that relates the integral over some local quantity (here the Ricci scalar) to a topological property of the manifold (here the number of holes). The more general version, called Atiyah–Singer index theorem, is one of the greatest mathematical discoveries from the 20^{th} century.

- Calculate first the expansion for the inverse metric

$$\gamma^{ij} = e^{-2\rho}\,\hat{\gamma}^{ij}_{(0)} + e^{-3\rho}\,\hat{\gamma}^{ij}_{(1)} + e^{-4\rho}\,\hat{\gamma}^{ij}_{(2)} + \dots$$

  and determine the coefficients $\hat{\gamma}^{ij}_{(n)}$ by requiring $\gamma^{ij}\gamma_{jk} = \delta^i_k$. It is convenient to use conventions such that all 2-dimensional indices are raised and lowered with the flat metric $\gamma^{(0)}_{ij} = \eta_{ij}$. Be careful with signs and be sure that you take into account all terms, particularly in $\hat{\gamma}^{ij}_{(2)}$. Obtaining the extrinsic curvature tensor is straightforward since we are in Gaussian normal coordinates (see the hint for the first exercise). The trace K is then obtained from multiplying the expansions $K_{ij}\gamma^{ij}$ up to the required order.