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Black Holes II — Exercise sheet 6

(16.1) **Mass and angular momentum of BTZ**
Take the BTZ metric ($\ell = 1$, $u = t+\phi$ and $v = t-\phi$ where $\phi \sim \phi+2\pi$)

$$\begin{aligned} \mathrm{d}s^2_{\text{BTZ}} &= \mathrm{d}\rho^2 + 4L\ \mathrm{d}u^2 + 4\bar{L}\ \mathrm{d}v^2 - \big(e^{2\rho} + 16L\bar{L}e^{-2\rho}\big)\ \mathrm{d}u\,\mathrm{d}v \\ &= \mathrm{d}\rho^2 + \big(e^{2\rho}\,\gamma^{(0)}_{ij} + \gamma^{(2)}_{ij} + \dots\big)\ \mathrm{d}x^i\,\mathrm{d}x^j \end{aligned}$$

and calculate the holographically renormalized Brown–York stress tensor ($G_N = 1$)

$$T^{\text{BY}}_{ij} = \frac{1}{8\pi}\Big(\gamma^{(2)}_{ij} - \gamma^{(0)}_{ij}\,\mathrm{Tr}\,\gamma^{(2)}\Big)$$

for all values of $m = L + \bar{L}$ and $j = L - \bar{L}$. Derive expressions for the conserved mass $M = \oint \mathrm{d}\phi\, T^{\text{BY}}_{tt}$ and angular momentum $J = \oint \mathrm{d}\phi\, T^{\text{BY}}_{t\phi}$. Which solution do you obtain for the special case $M = -1/8$, $J = 0$?

(16.2) **BTZ and AdS**
Take global AdS

$$\mathrm{d}s^2 = -\cosh^2\!\rho\ \mathrm{d}t^2 + \sinh^2\!\rho\ \mathrm{d}\phi^2 + \mathrm{d}\rho^2$$

and perform the double Wick rotation $t \to it$, $\phi \to i\phi$ together with the shift $\rho \to \rho + i\pi/2$. Show that the resulting line-element can be mapped to the BTZ metric

$$\mathrm{d}s^2_{\text{BTZ}} = \mathrm{d}\hat{\rho}^2 + 4L\ \mathrm{d}u^2 + 4\bar{L}\ \mathrm{d}v^2 - \big(e^{2\hat{\rho}} + 16L\bar{L}e^{-2\hat{\rho}}\big)\ \mathrm{d}u\,\mathrm{d}v$$

upon appropriately transforming the coordinates u, v, $\hat{\rho}$ to t, ϕ, ρ. Discuss why this mapping does not contradict the fact that BTZ and AdS are globally not diffeomorphic to each other.

(16.3) **Central charge in Virasoro algebra**
Consider the vector fields $L_n = -z^{n+1}\partial_z$, which generate the Witt algebra

$$[L_n,\, L_m] = (n-m)\, L_{n+m}$$

Show that the central extension to the Virasoro algebra ($c,\ \tilde{c} \in \mathbb{R}$)

$$[L_n,\, L_m] = (n-m)\, L_{n+m} + \frac{c}{12}\, n(n^2 - \tilde{c})\, \delta_{n+m,\,0}$$

is compatible with the Jacobi identities $[[L_n,\, L_m],\, L_k] + \mathrm{cycl}(n,m,k) = 0$. Moreover, show that we can always achieve $\tilde{c} = 1$ by an appropriate shift of the generator $L_0 \to L_0 + \ell_0$, where ℓ_0 is some c-number. Show finally that only for this choice the sub-algebra generated by L_{-1}, L_0, L_1 has no central extension. [You get bonus points if you show that the central extension above is unique.]

These exercises are due on May 3rd 2012.

Hints:

- Read off the Fefferman–Graham expansion matrices $\gamma_{ij}^{(0)}$ and $\gamma_{ij}^{(2)}$. Insert them into the result for the Brown–York stress tensor and give explicitly expressions for T_{uu}, T_{vv} and T_{uv}, either in terms of (m, j) or in terms of $(L, \bar{L})$. Then calculate the conserved charges using the coordinate transformation $(u, v) \to (t, \phi)$ and the standard tensor transformation law. To study the special case $M = -1/8$ and $J = 0$ just insert the corresponding values for $L, \bar{L}$ into the line-element. After a suitable shift $\rho \to \rho + \rho_0$ and upon expressing (u, v) in terms of (t, ϕ) it should be manifest which spacetime this is [it has a lot of Killing vectors].

- After performing the double Wick rotation and the imaginary shift in ρ use the identities $\sinh(\rho + i\pi/2) = i \cosh \rho$ and $\cosh(\rho + i\pi/2) = i \sinh \rho$. Then work backwards from the BTZ line-element, using the linear transformation $u = a\phi + bt$, $v = c\phi + dt$. Demand that the resulting line-element is equal to the one obtained from Wick rotation. The vanishing of the mixed term $\mathrm{d}t\,\mathrm{d}\phi$ gives you two conditions that fix e.g. d and c in terms of the other constants. Consider then the $\mathrm{d}\phi^2$ and $\mathrm{d}t^2$ terms and complete the square, obtaining expressions of the form $\#(e^{\hat{\rho}} \pm \# e^{-\hat{\rho}})^2$, where $\#$ are constants that you have to determine. Perform a real shift $\hat{\rho} = \rho + \rho_0$ and fix ρ_0 such that the complete squares are proportional to $\cosh \rho$ and $\sinh \rho$ (depending on the sign). Finally, fix a and b suitably. Regarding the discussion, it is sufficient to discuss the differences between the geometries before and after the Wick rotation, so you can do this part without using the longer calculation that shows equivalence to BTZ. Focus in particular on the range of the coordinates and periodicity properties!

- Just follow the instructions of the exercise: check first all Jacobi identities and then perform the shift of L_0 to achieve $\tilde{c} = 1$. It is straightforward to check for which generators the Virasoro algebra has vanishing central term. [The bonus part is less trivial.]