Daniel Grumiller May 3rd 2012

Black Holes II — Exercise sheet 7

(17.1) **Boundary conditions in AdS_3 quantum gravity**
Given some boundary conditions for the metric, the asymptotic symmetry group consists of those diffeomorphisms that preserve these boundary conditions (modulo trivial diffeomorphisms). Consider the Brown–Henneaux boundary conditions [J. D. Brown and M. Henneaux, *Commun. Math. Phys.* **104** (1986) 207] near $\rho \to \infty$,

$$g_{\alpha\beta} = \begin{pmatrix} g_{++} = \mathcal{O}(1) & g_{+-} = -\frac{1}{2}\,e^{2\rho} + \mathcal{O}(1) & g_{+\rho} = \mathcal{O}(e^{-2\rho}) \\ g_{-+} = g_{+-} & g_{--} = \mathcal{O}(1) & g_{-\rho} = \mathcal{O}(e^{-2\rho}) \\ g_{\rho+} = g_{+\rho} & g_{\rho-} = g_{-\rho} & g_{\rho\rho} = 1 + \mathcal{O}(e^{-2\rho}) \end{pmatrix}$$

where the total metric $g = \bar{g} + h$ consists of an asymptotic AdS_3 background

$$\bar{g}_{\alpha\beta}\, \mathrm{d}x^\alpha\, \mathrm{d}x^\beta = \mathrm{d}\rho^2 - e^{2\rho}\, \mathrm{d}x^+\, \mathrm{d}x^-$$

and of fluctuations h that fall off near $\rho \to \infty$ according to the boundary conditions above. Show that boundary condition-preserving transformations $\mathcal{L}_\xi g = \mathcal{O}(h)$ are given by diffeomorphisms generated by a vector field ξ of the form

$$\begin{aligned} \xi^+ &= \varepsilon^+(x^+) + \frac{1}{2}\, e^{-2\rho}\, \partial_-^2 \varepsilon^-(x^-) + \mathcal{O}(e^{-4\rho}) \\ \xi^- &= \varepsilon^-(x^-) + \frac{1}{2}\, e^{-2\rho}\, \partial_+^2 \varepsilon^+(x^+) + \mathcal{O}(e^{-4\rho}) \\ \xi^\rho &= -\frac{1}{2}\, \big(\partial_+ \varepsilon^+(x^+) + \partial_- \varepsilon^-(x^-)\big) + \mathcal{O}(e^{-2\rho}) \end{aligned}$$

where $\varepsilon^\pm$ are arbitrary functions of their arguments.

(17.2) **Anomalous transformation of stress tensor**
Take some asymptotic AdS line-element (you can think of BTZ),

$$\mathrm{d}s^2 = \mathrm{d}\rho^2 + 4L(x^+)\,(\mathrm{d}x^+)^2 + 4\bar{L}(x^-)\,(\mathrm{d}x^-)^2 - e^{2\rho}\, \mathrm{d}x^+\, \mathrm{d}x^- + \mathcal{O}(e^{-2\rho})$$

and determine the Lie-variation of L and $\bar{L}$ generated by a vector field ξ like in exercise (17.1).

(17.3) **Local triviality vs. global non-triviality**
Discuss how/why/under which conditions it is possible that a physical theory has no local propagating physical degree of freedom and nevertheless is non-trivial. Give at least one specific example.

These exercises are due on May 31st 2012.

Hints:

- Recall that a diffeomorphism generated by a vector field ξ acts on the metric via the Lie derivative

$$\mathcal{L}_\xi \, g_{\mu\nu} = \xi^\sigma \partial_\sigma g_{\mu\nu} + g_{\mu\sigma}\partial_\nu \xi^\sigma + g_{\nu\sigma}\partial_\mu \xi^\sigma$$

  Check now that the diffeomorphisms generated by the vector field ξ given at the end of (17.1) preserve the Brown–Henneaux boundary conditions. As an example here is how you check that the $++$-component is ok:

$$\mathcal{L}_\xi g_{++} = \xi^\mu \partial_\mu g_{++} + 2g_{+\mu}\partial_+\xi^\mu = 2\bar{g}_{+-}\partial_+\xi^- + \mathcal{O}(1) = \mathcal{O}(1) = \mathcal{O}(h_{++})$$

  [Note: the asymptotic symmetry algebra is generated by the functions $\varepsilon^\pm(x^\pm)$ and leads to the Virasoro algebra. To show this qualitatively one just has to consider the exercises (16.3) and (17.2). To really calculate the value of the left and right central charges, $c_L = c_R = 3\ell/(2G_N) = 3/2$, one has to perform a canonical analysis.]

- To obtain the variation δL note that $4\delta L = \delta g_{++}$, where δg is the Lie variation of the metric along the vector field ξ (see the hint above). The final result that you find should be of the form

$$\delta L = \varepsilon^+ \partial_+ L + 2L\partial_+\varepsilon^+ + \#\, \partial_+^3 \varepsilon^+$$

  where # is some number (that you should determine). Analogous considerations apply to the other chirality $\bar{L}$. [Note: the first two terms in the transformation law of L are the expected transformation law for a conformal primary of conformal dimension 2; the last term is an anomalous contribution and matches with the central term in the Virasoro algebra.]

- A possible example is Einstein gravity in three dimensions, but there are many other examples. Recall what you have learned in the lectures about three-dimensional Einstein gravity.