Daniel Grumiller May 3^{rd} 2012

Black Holes II — Exercise sheet 9

(19.1) **First Law**
The first law of thermodynamics states

$$\mathrm{d}E(S,\,J) = T\,\mathrm{d}S + \Omega\,\mathrm{d}J$$

where the $\Omega\,\mathrm{d}J$-term is a work term caused by the presence of an angular potential Ω and angular momentum J. Show that for BTZ black holes the first law of black hole mechanics is valid, which looks *precisely* as the first law of thermodynamics.

(19.2) **Second Law**
The second law of black hole mechanics (the Hawking area theorem) states that under "reasonable" assumptions[1] the area A of a black hole event horizon is a monotonically increasing function of time.

$$\mathrm{d}A \geq 0$$

By comparison with (18.2) A is proportional to entropy, which in turn counts the number of black hole microstates. [See the note at the end of the hints for exercise (18.2).] Let us now go back to 4 dimensions, where qualitatively the same happens as in 3 dimensions. Calculate A for a solar mass Schwarzschild black hole and provide an estimate for the number of microstates of such a black hole. Discuss (either colloquially or with formulas) what happens when you take a box filled with photons of a certain temperature, energy and entropy and drop it into the black hole.

(19.3) **Third Law**
The third law of black hole mechanics states that physical processes that lead to vanishing surface gravity

$$\kappa \to 0$$

are not possible in finite time. Discuss for a Schwarzschild black hole how you could attempt to violate the third law and why such attempts do not work. Generalize this discussion to Reissner–Nordström black holes.

These exercises are due on May 31^{st} 2012.

[1] Some of these assumptions, in particular any energy condition that one imposes, are generically violated by quantum fields. But classically they are indeed reasonable.

Hints:

- The internal energy is expressed as a function of the extensive quantities S and J. Use a double Legendre-transformation, $E(S, J) = F(T, \Omega) + TS + \Omega J$, to relate internal energy to the free energy given in exercise (18.2). Then remember that $M = L + \bar{L}$ and $J = L - \bar{L}$ and use the relations between various quantities given in the hints for the bonus part of exercise (18.2). At intermediate steps you could express everything in terms of $r_\pm$. If you do this you should get the intermediate result $E = (r_+^2 + r_-^2)/8$. At the end express $r_\pm$ in terms of S and J. Check finally if it is true that $\partial E/\partial S = T$ and $\partial E/\partial J = \Omega$.

- Calculate A in natural units and recall how the number of microstates scales with entropy. For the colloquial discussion compare with exercise (8.3).

- Remember how surface gravity is related to mass and consider what you would have to do with the mass of a Schwarzschild black hole in order to make surface gravity vanish. For the Reissner–Nordström case [see exercise (9.1) for the corresponding line-element] start with a sub-extremal black hole $|Q| < M$ and try to make it extremal by dropping charged particles (which you are allowed to model for simplicity as spherical shells with a certain mass and charge) into it. Note that the particle only falls into the black hole if gravitational attraction overcomes the electrostatic repulsion. There are various ways you can do this calculation (see e.g. the lecture notes by Townsend, `gr-qc/9707012`, section 3); perhaps the simplest one is to use just Coulomb's law and Newton's gravity law (though you must address under which conditions you can trust the latter).

Historical note: together with the zeroth law (surface gravity is constant on the event horizon of stationary black holes) the four laws of black hole mechanics are in one-to-one correspondence with the four laws of thermodynamics. This is not a coincidence, but a rather deep result that relates black hole physics and gravity with thermodynamics and statistical mechanics, and has culminated in the holographic principle, realized explicitly in the AdS/CFT correspondence and other gauge/gravity correspondences. A nice summary of the four laws of black hole mechanics in four dimensions is provided in J. M. Bardeen, B. Carter and S. W. Hawking, "The Four laws of black hole mechanics," *Commun. Math. Phys.* **31** (1973) 161. [free version: click here or type `http://projecteuclid.org/DPubS?service=UI&version=1.0&verb=Display&handle=euclid.cmp/1103858973`]