Black Holes II — Exercise sheet 10

(20.1) **Linearized fluctuations around AdS**
Split the metric into vacuum plus fluctuations, $g = \bar{g} + h$, where $\bar{g}$ is the global AdS metric and h some fluctuation. In the lectures we showed that the linearized Einstein tensor (with unit AdS radius) is given by

$$\mathcal{G}(h)_{\mu\nu} = \frac{1}{2}\big(- \bar{\nabla}^2 h_{\mu\nu} - \bar{\nabla}_\mu \bar{\nabla}_\nu h + \bar{\nabla}_\nu \bar{\nabla}^\sigma h_{\sigma\mu} + \bar{\nabla}_\mu \bar{\nabla}^\sigma h_{\sigma\nu} - 2h_{\mu\nu} \\ - \bar{g}_{\mu\nu}(\bar{\nabla}_\sigma \bar{\nabla}_\tau h^{\sigma\tau} - \bar{\nabla}^2 h)\big)$$

where $h = h^\mu{}_\mu = \bar{g}^{\mu\nu} h_{\mu\nu}$. Split the fluctuations into transverse-traceless, "trace" and gauge-part

$$h_{\mu\nu} = h^{TT}_{\mu\nu} + \frac{1}{3}\,\bar{g}_{\mu\nu}\tilde{h} + \bar{\nabla}_\mu \xi_\nu + \bar{\nabla}_\nu \xi_\mu$$

where ξ^μ is some vector field, $\tilde{h}$ some scalar field and $\bar{g}^{\mu\nu} h^{TT}_{\mu\nu} = \bar{\nabla}^\mu h^{TT}_{\mu\nu} = 0$. Show that the linearized operator $\mathcal{G}$ vanishes when acting on gauge modes, $\mathcal{G}(\bar{\nabla}\xi)_{\mu\nu} = 0$ and write down simple expression for the linearized operator acting on transverse-traceless modes $\mathcal{G}(h^{TT})_{\mu\nu} = 0$ and on "trace" modes $\mathcal{G}(\bar{g}\tilde{h})_{\mu\nu} = 0$. [Bonus: discuss why I put "trace" under quotation marks.]

(20.2) **1-loop partition function**
Show that for 3-dimensional Einstein gravity around an AdS vacuum the 1-loop path integral over the quadratic metric fluctuations leads to the 1-loop partition function

$$Z^{(1)} = \Big[\det\big(-\bar{\nabla}^2 - 2\big)_T \det\big(-\bar{\nabla}^2 + 3\big)_S\Big]^{-1/2} \times Z_{\text{gh}}$$

where $Z_{\text{gh}} = \big[\det\big(-\bar{\nabla}^2 + 2\big)_V \det\big(-\bar{\nabla}^2 + 3\big)_S\big]^{1/2}$ is the ghost-determinant and the subscripts T, S, V denote, respectively, transverse traceless tensor modes, scalar modes and transverse vector modes.

(20.3) **Quantum gravity partition function**
Consider 3-dimensional Euklidean Einstein gravity. Write down the full quantum mechanical partition function and split it in the following way

$$Z = \sum_n \prod_{m=0}^{\infty} Z_n^{(m)}$$

where n labels the sum over all non-perturbative classical backgrounds (AdS_3 and BTZ black holes) and m denotes the number of quantum loops, i.e., $Z_n^{(0)}$ is the classical partition function of the n-th background, $Z_n^{(1)}$ is the 1-loop partition function etc. One can argue that all $Z_n^{(m)} = 1$ for $m \geq 2$, i.e., the theory is 1-loop exact. Use this argument to find a representation of the full quantum gravity partition function that is as simple as possible.

These exercises are due on June 14$^{\text{th}}$ 2012.

Hints:

- Follow the instructions. The first part with the gauge modes can be solved in two ways: a clever, rather short way and a lengthy but straightforward way. I give only hints concerning the latter. Remember that the covariant derivatives $\bar{\nabla}$ do not commute when acting on vector or tensor fields! Use the relation

$$[\bar{\nabla}_\mu, \bar{\nabla}_\nu]\xi^\sigma = \bar{R}^\sigma{}_{\tau\mu\nu}\xi^\tau$$

  and similarly relations for tensors as well as the fact that for AdS with unity AdS radius the Riemann-tensor simplifes to

$$\bar{R}_{\alpha\beta\gamma\delta} = \bar{g}_{\alpha\delta}\bar{g}_{\beta\gamma} - \bar{g}_{\alpha\gamma}\bar{g}_{\beta\delta}$$

  Related useful formulas are $\bar{R}_{\mu\nu} = -2\bar{g}_{\mu\nu}$ and $\bar{R} = -6$. The second and third parts regarding transverse-traceless and trace modes is straightforward, so you should need no further hint.

- In the lectures you learned that the 1-loop partition function is given by the expression

$$Z^{(1)} = \int \mathcal{D}h^{TT}_{\mu\nu}\mathcal{D}\tilde{h}\, \exp\Big(- \int \mathrm{d}^3x\sqrt{|\bar{g}|}h^{\mu\nu}\mathcal{G}(h)_{\mu\nu}\Big) \times Z_{\text{gh}}$$

  Use the results from exercise (20.1) to express $\mathcal{G}(h)_{\mu\nu}$ conveniently in terms of h^{TT} and $\tilde{h}$ (note that the ξ-contribution must drop out according to the previous exercise). Finally, use the result that the path integral over some bosonic operator leads to the determinant of this operator to the power $-1/2$ (remember Gaussian integrals!)

$$\int \mathcal{D}\Phi \exp\Big(- \int \mathrm{d}^3x\sqrt{|\bar{g}|}\, \Phi\mathcal{O}\Phi\Big) = \Big[\det \mathcal{O}\Big]^{-1/2}$$

- You can use the results of exercises (20.2), (18.2) and (18.1). There is not much you have to calculate. Or perhaps there is. I leave this to you. The most convenient way to represent the partition function is via a path integral representation.