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Black Holes II — Exercise sheet 1

(11.1) **Coordinate transformation reminder**
Take the Minkowski metric in standard spherical coordinates t, r, θ, ϕ. Perform successively the following two coordinate transformations: $u = t - r$, $v = t + r$ and $\tilde{U} = \arctan u$, $\tilde{V} = \arctan v$. Write down the Minkowski line-element in the coordinates $\tilde{U}, \tilde{V}, \theta, \phi$ and pull out an overall factor $1/(\cos\tilde{U}\,\cos\tilde{V})^2$. What is the range of the coordinates $\tilde{U}$ and $\tilde{V}$? Is that range an open or a closed set? What happens at $\tilde{U} = \tilde{V}$? What happens for $|\tilde{U}| \to \pi/2$ and/or $|\tilde{V}| \to \pi/2$?

(11.2) **Killing horizon reminder**
The line-element

$$\mathrm{d}s^2 = g_{\mu\nu}\,\mathrm{d}x^\mu\,\mathrm{d}x^\nu = -K(r)\,\mathrm{d}t^2 + \frac{\mathrm{d}r^2}{K(r)} + r^2\,\mathrm{d}\Omega^2_{S^{D-2}}$$

describes various interesting spacetimes, depending on the function $K(r)$ and the spacetime dimension D (here $\mathrm{d}\Omega^2_{S^{D-2}}$ is the line-element of the round $D-2$ sphere). Suppose $D = 4$ and

$$K(r) = 1 - \frac{2M}{r} + \frac{Q^2}{r^2} - ar - \Lambda r^2$$

where $\Lambda, a, Q \in \mathbb{R}$ and $M \in \mathbb{R}^+$. How many Killing horizons can arise at most in this spacetime? What kind of special cases allow for at least one extremal Killing horizon? Suppose that

(a) Λ
(b) a
(c) M

is very large and positive so that you can neglect all other terms in the Killing norm except for the 1. What kind of spacetime (and what kind of Killing horizon) do you get in each of these cases?

(11.3) **Black hole mind bender**
If gravitons cannot escape from inside a black hole and the gravitational field consists of gravitons then how can an observer outside the black hole feel its gravitational field?

These exercises are due on March 11$^{\text{th}}$ 2014.

Hints/comments:

- The Minkowski metric in spherical coordinates is given by

$$\mathrm{d}s^2 = -\,\mathrm{d}t^2 + \mathrm{d}r^2 + r^2\ \mathrm{d}\Omega^2_{S^2}$$

with $\mathrm{d}\Omega^2_{S^2} = \mathrm{d}\theta^2 + \sin^2\theta\ \mathrm{d}\phi^2$. For the actual exercise just perform the two coordinate transformations, which is most easily done using the coordinate differentials (e.g. $\mathrm{d}u = \mathrm{d}t - \mathrm{d}r$), and keeping track of the ranges of definitions. In the original coordinate system above we have $t \in (-\infty, \infty)$, $r \in [0, \infty)$, $\theta \in [0, \pi]$, $\phi \in [0, 2\pi)$. The trigonometric formula

$$\tan\alpha - \tan\beta = \frac{\sin(\alpha - \beta)}{\cos\alpha\ \cos\beta}$$

may come in handy.

- You probably realized immediately that the metric provided in this exercise exhibits a Killing vector ∂_t whose norm is determined by the function $K(r)$. Presumably you also remember that a Killing horizon corresponds to a locus in spacetime where the norm of that Killing vector vanishes, so that it constitutes the normal vector to a null hypersurface. Now it is a matter of simple algebra and counting to determine the maximal number of Killing horizons. Let me remind you that an extremal Killing horizon is a Killing horizon whose surface gravity vanishes. We derived a very useful formula for surface gravity in exercise (8.2) that allows you to find a simple condition for extremality. Note that you do not have to find *all* possible ways of creating an extremal Killing horizon, but only give *some* examples. Concerning the last part of the exercise, you may wish to look back at exercises (9.1), (7.3) and (4.2). If you did not visit the lectures Black Holes I you can find the exercises (1.1)-(10.3) as PDF-files at `http://quark.itp.tuwien.ac.at/~grumil/teaching.shtml` (scroll down a bit).

- This is a question frequently asked by students. I thought it was a nice change if it is asked by the lecturer instead ;-) The answer is simple, and yet...