

Black Holes II — Alternative exercise sheet 5

(15.1) Field equations for spherically symmetric charged BHs

Requiring spherical symmetry at the level of the action reduces the Einstein–Hilbert–Maxwell action in 4D to a specific dilaton gravity–Maxwell action in 2D (to reduce clutter we set $\kappa = 1$):

$$S^{\text{RNBH}} = \int d^2x \sqrt{-g} \left[XR + \frac{(\nabla X)^2}{2X} + 1 \right] - \int d^2x \sqrt{-g} X F_{\mu\nu} F^{\mu\nu}$$

Derive the equations of motion for metric g , dilaton X and gauge-field A_μ (as usual $F_{\mu\nu} = \nabla_\mu A_\nu - \nabla_\nu A_\mu = \partial_\mu A_\nu - \partial_\nu A_\mu$). Solve for $F_{\mu\nu}$ using the field equations for A_μ . Interpret this result from a 4D perspective. What is the physical role of the constant of motion appearing in the solution of the field equations for A_μ ?

(15.2) Deriving the Reissner-Nordström solution

Take the equations of motion derived in alternative exercise (15.1) and find their most general solution for the metric g , non-constant dilaton X and gauge-field A_μ . Oxidize your result to 4D and write down the line-element in Schwarzschild coordinates. This line-element is the Reissner-Nordström solution. Finally, find all constant dilaton vacua. Are these vacuum geometries $\text{AdS}_2 \times S^2$, $\text{Minkowski} \times S^2$ or $\text{dS}_2 \times S^2$?

(15.3) Asymptotic quasi-normal modes in 2D dilaton gravity

Take the dilaton gravity action from alternative exercise (14.1) and couple it non-minimally to a massless scalar field ϕ

$$S = S^{\text{2DG}} + S^{\text{mat}} \quad S^{\text{mat}} = -\frac{1}{2} \int d^2x \sqrt{-g} X^p (\nabla\phi)^2$$

with constant p . Using the monodromy approach by Motl and Neitzke ([gr-qc/0212096](#) and [hep-th/0301173](#)), the quasi-normal mode spectrum of this model was analyzed in [gr-qc/0408042](#) in the limit of large damping. For the complex frequency ω the asymptotic relation

$$e^{\omega/T_H} = -(1 + 2 \cos(\pi(1-p)))$$

was found, where T_H is the Hawking temperature of the black hole. Asymptotic means that the imaginary part of ω/T_H is large and positive. Consider the minimally coupled case ($p = 0$) and the Schwarzschild case ($p = 1$) and derive formulas for the real and imaginary parts of ω/T_H . Compare the $p = 1$ case with the (computer-) experimental results for the Schwarzschild black hole by Nollert (*Phys. Rev.* **D47** (1993) 5253) and Andersson (*Class. Quant. Grav.* **L10** (1993) 61) who found the asymptotic formula

$$\omega M \approx 0.0437 + \frac{i}{4} \left(n + \frac{1}{2} \right)$$

Does the 2D dilaton gravity formulation of the Schwarzschild black hole lead to the correct asymptotic spectrum for quasi-normal modes?

These exercises are due on April 8th 2014.

Hints:

- The field equations for dilaton and metric can be derived essentially in the same way as for alternative exercise (14.1) — see the hints there. For consistency your field equations in the limit $A_\mu \rightarrow 0$ must coincide with the ones derived in the alternative exercise (14.1) for the special case $U(X) = -\frac{1}{2X}$ and $V(X) = -\frac{1}{2}$. The field equation for the gauge field is straightforward. You should obtain

$$\nabla_\mu (X F^{\mu\nu}) = \varepsilon^{\mu\nu} \partial_\mu (X f) = 0$$

where the first equality exploits the fact that any anti-symmetric tensor $F^{\mu\nu}$ in 2D can be written as $F^{\mu\nu} = \varepsilon^{\mu\nu} f$, where f is a scalar field and $\varepsilon^{\mu\nu}$ the ε -tensor. It is straightforward to solve the field equations above for f in terms of X and an integration constant. Regarding the 4D interpretation remember how the dilaton field X is related to the standard radial coordinate r and compare with the Coulomb solution.

- If you did not do alternative exercise (15.1) then you need to know the field equations. The one for A_μ is provided in the hint above. The one for the dilaton reads

$$R + \frac{(\nabla X)^2}{2X^2} - \frac{\nabla^2 X}{X} - F^2 = 0$$

where $F^2 = F_{\mu\nu} F^{\mu\nu}$. Variation with respect to the metric yields

$$\nabla_\mu \nabla_\nu X - g_{\mu\nu} \nabla^2 X - \frac{(\nabla_\mu X)(\nabla_\nu X)}{2X} + g_{\mu\nu} \frac{(\nabla X)^2}{4X} + \frac{1}{2} g_{\mu\nu} = -2X T_{\mu\nu}$$

with

$$T_{\mu\nu} = F_{\mu\alpha} F_\nu^\alpha - \frac{1}{4} g_{\mu\nu} F^2 = -\frac{1}{2} g_{\mu\nu} f^2$$

In the last equality I have used $F_{\mu\nu} = \varepsilon_{\mu\nu} f$ as well as the 2D identity $\varepsilon_{\mu\alpha} \varepsilon^{\alpha\nu} = g_{\mu\nu}$ (for Minkowski signature). Exploit now the fact that you can introduce an effective 2D dilaton gravity model with potentials

$$U^{\text{eff}} = U = -\frac{1}{2X} \quad V^{\text{eff}} = -\frac{1}{2} + \frac{q^2}{4X}$$

where q is the constant of motion appearing in the dual field strength $f = q/(2X)$. This trick allows you to take advantage of the results derived in the lectures for general 2D dilaton gravity solutions with non-constant dilaton. For the oxidation remember that the 4D line element is determined from the 2D metric $g_{\alpha\beta}$ and the dilaton X as follows:

$$ds_{(4)}^2 = g_{\alpha\beta} dx^\alpha dx^\beta + 2X d\Omega_{S^2}^2$$

For the constant dilaton vacua use $X = \text{const.}$ as early as possible. Exploit that the Ricci scalar R uniquely determines the Riemann tensor in 2D — if you know e.g. that R is constant spacetime can only be de Sitter, Minkowski or Anti-de Sitter, depending on the sign of R .

- What is T_H in terms of M for the Schwarzschild black hole? (We motivated this result last semester, see exercise (9.3), and we shall derive it from scratch this semester.) Very short exercise!