Daniel Grumiller April 8th 2014

Black Holes II — Exercise sheet 6

(16.1) **Generalized Fefferman–Graham expansion** [e.g. `arXiv:1310.0819`]
Consider an asymptotically AdS metric in Gaussian normal coordinates

$$\mathrm{d}s^2 = \mathrm{d}\rho^2 + \gamma_{ij}\,\mathrm{d}x^i\,\mathrm{d}x^j$$

with the following asymptotic expansion in the limit of large ρ

$$\gamma_{ij} = e^{2\rho}\gamma_{ij}^{(0)} + e^{\rho}\gamma_{ij}^{(1)} + \gamma_{ij}^{(2)} + \dots$$

where $\gamma_{ij}^{(0)} = \eta_{ij}$ is the flat Minkowski metric. Calculate the asymptotic expansion for extrinsic curvature K_{ij} and its trace $K = K_{ij}\gamma^{ij}$. It is sufficient to keep only the three leading terms in these expansions (note that some of the terms may vanish; in that case you need *not* go to even higher order in the expansion).

(16.2) **Mass and angular momentum of BTZ** [`hep-th/9204099`]
Take the BTZ metric ($\ell = 1$, $u = t+\phi$ and $v = t-\phi$ where $\phi \sim \phi+2\pi$)

$$\begin{aligned}\mathrm{d}s^2_{\text{BTZ}} &= \mathrm{d}\rho^2 + 4L\,\mathrm{d}u^2 + 4\bar{L}\,\mathrm{d}v^2 - \big(e^{2\rho} + 16L\bar{L}e^{-2\rho}\big)\,\mathrm{d}u\,\mathrm{d}v \\ &= \mathrm{d}\rho^2 + \big(e^{2\rho}\,\gamma_{ij}^{(0)} + \gamma_{ij}^{(2)} + \dots\big)\,\mathrm{d}x^i\,\mathrm{d}x^j\end{aligned}$$

and calculate the holographically renormalized Brown–York stress tensor ($G_N = 1$)

$$T_{ij}^{\text{BY}} = \frac{1}{8\pi}\Big(\gamma_{ij}^{(2)} - \gamma_{ij}^{(0)}\,\mathrm{Tr}\,\gamma^{(2)}\Big)$$

for all values of $m = L + \bar{L}$ and $j = L - \bar{L}$. Derive expressions for the conserved mass $M = \oint \mathrm{d}\phi\, T_{tt}^{\text{BY}}$ and angular momentum $J = \oint \mathrm{d}\phi\, T_{t\phi}^{\text{BY}}$. Which solution do you obtain for the special case $M = -1/8$, $J = 0$?

(16.3) **BTZ and AdS$_3$** [`gr-qc/9302012`]
Take global AdS

$$\mathrm{d}s^2 = -\cosh^2\!\rho\;\mathrm{d}t^2 + \sinh^2\!\rho\;\mathrm{d}\phi^2 + \mathrm{d}\rho^2$$

and perform the double Wick rotation $t \to it$, $\phi \to i\phi$ together with the shift $\rho \to \rho + i\pi/2$. Show that the resulting line-element can be mapped to the BTZ metric

$$\mathrm{d}s^2_{\text{BTZ}} = \mathrm{d}\hat{\rho}^2 + 4L\,\mathrm{d}u^2 + 4\bar{L}\,\mathrm{d}v^2 - \big(e^{2\hat{\rho}} + 16L\bar{L}e^{-2\hat{\rho}}\big)\,\mathrm{d}u\,\mathrm{d}v$$

upon appropriately transforming the coordinates u, v, $\hat{\rho}$ to t, ϕ, ρ. Discuss why this mapping does not contradict the fact that BTZ and AdS are globally not diffeomorphic to each other.

These exercises are due on April 29th 2014.

Hints:

- Calculate first the expansion for the inverse metric

$$\gamma^{ij} = e^{-2\rho}\,\hat{\gamma}^{ij}_{(0)} + e^{-3\rho}\,\hat{\gamma}^{ij}_{(1)} + e^{-4\rho}\,\hat{\gamma}^{ij}_{(2)} + \dots$$

  and determine the coefficients $\hat{\gamma}^{ij}_{(n)}$ by requiring $\gamma^{ij}\gamma_{jk} = \delta^i_k$. It is convenient to use conventions such that all boundary indices are raised and lowered with the flat metric $\gamma^{(0)}_{ij} = \eta_{ij}$. Be careful with signs and be sure that you take into account all terms, particularly in $\hat{\gamma}^{ij}_{(2)}$. Obtaining the extrinsic curvature tensor is straightforward since we are in Gaussian normal coordinates (see your lecture notes or re-derive the result for K_{ij} in Gaussian normal coordinates). The trace K is then obtained from multiplying the expansions $K_{ij}\gamma^{ij}$ up to the required order.

- Read off the Fefferman–Graham expansion matrices $\gamma^{(0)}_{ij}$ and $\gamma^{(2)}_{ij}$. Insert them into the result for the Brown–York stress tensor and give explicitly expressions for T_{uu}, T_{vv} and T_{uv}, either in terms of (m, j) or in terms of $(L, \bar{L})$. Then calculate the conserved charges using the coordinate transformation $(u, v) \to (t, \phi)$ and the standard tensor transformation law. To study the special case $M = -1/8$ and $J = 0$ just insert the corresponding values for $L, \bar{L}$ into the line-element. After a suitable shift $\rho \to \rho + \rho_0$ and upon expressing (u, v) in terms of (t, ϕ) it should be manifest which spacetime this is [it has a lot of Killing vectors].

- After performing the double Wick rotation and the imaginary shift in ρ use the identities $\sinh(\rho + i\pi/2) = i\cosh\rho$ and $\cosh(\rho + i\pi/2) = i\sinh\rho$. Then work backwards from the BTZ line-element, using the linear transformation $u = a\phi + bt$, $v = c\phi + dt$. Demand that the resulting line-element is equal to the one obtained from Wick rotation. The vanishing of the mixed term $\mathrm{d}t\,\mathrm{d}\phi$ gives you two conditions that fix e.g. d and c in terms of the other constants. Consider then the $\mathrm{d}\phi^2$ and $\mathrm{d}t^2$ terms and complete the square, obtaining expressions of the form $\#(e^{\hat{\rho}} \pm \# e^{-\hat{\rho}})^2$, where $\#$ are constants that you have to determine. Perform a real shift $\hat{\rho} = \rho + \rho_0$ and fix ρ_0 such that the complete squares are proportional to $\cosh\rho$ and $\sinh\rho$ (depending on the sign). Finally, fix a and b suitably. Regarding the discussion, it is sufficient to discuss the differences between the geometries before and after the Wick rotation, so you can do this part without using the longer calculation that shows equivalence to BTZ. Focus in particular on the range of the coordinates and periodicity properties!