Daniel Grumiller May 13th 2014

Black Holes II — Exercise sheet 8

(18.1) **On-shell action and free energy of BTZ black holes**
Take the holographically renormalized 3-dimensional Euclidean Einstein gravity action ($G_N = \ell = 1$)

$$\Gamma = -\frac{1}{16\pi}\int_M \mathrm{d}^3x\sqrt{g}\,\big(R+2\big) - \frac{1}{8\pi}\int_{\partial M}\mathrm{d}^2x\sqrt{\gamma}\,\big(K-1\big)$$

and evaluate the saddle point approximation of the Euclidean path integral

$$Z = e^{-\Gamma[g_{\mathrm{cl}}]}\times Z_{\mathrm{qu}}$$

where g_{cl} is some solution of the classical equations of motion, like global AdS or BTZ black holes. Ignore the quantum fluctuations encoded by the quantum partition function $Z_{\mathrm{qu}} \approx 1$. Then calculate the Helmholtz free energy for all (for simplicity non-rotating) BTZ black holes and for global AdS

$$F = -T\,\ln Z$$

where T is the Hawking temperature (or inverse period of Euclidean time) and represent it in terms of mass M and/or temperature T. [Bonus: consider the generalization to rotating BTZ black holes.]

(18.2) **Entropy of BTZ black holes and Bekenstein–Hawking law**
The free energy of rotating BTZ black holes is given by

$$F(T,\,\Omega) = -\frac{\pi^2}{2}\cdot\frac{T^2}{1-\Omega^2}$$

where $T = (r_+^2 - r_-^2)/(2\pi r_+)$ is the temperature and $\Omega = r_-/r_+$ is the angular potential (obeying $|\Omega| \leq 1$). [As usual we have set $G_N = \ell = 1$.] Calculate the entropy of all BTZ black holes and investigate whether the Bekenstein–Hawking entropy law holds.

(18.3) **Information loss problem in condensed matter physics**
Consider a piece of coal at zero temperature and a laser beam (a pure quantum state with some finite energy and entropy) in vacuum as initial state. Provided the laser beam is directed toward the piece of coal it will eventually be absorbed and scattered by the coal. In this (complicated) process the coal will heat up a little bit. Suppose that the coal is a nearly perfect black body. Then the final state will be the scattered pure radiation and the outgoing thermal black-body radiation emitted by the piece of coal. Thus, we appear to have an evolution of a pure initial state into a final state that is not pure. Information is lost, similar to what happens in the case of an evaporating black hole. How is this information loss problem resolved in condensed matter physics?

These exercises are due on May 20th 2014.

Hints:

- There are many equivalent ways to derive the on-shell action. A simple approach invokes the line-element $\mathrm{d}s^2 = \mathrm{d}\rho^2 + \sinh^2\!\rho\,\mathrm{d}\tau^2 + \cosh^2\!\rho\,\mathrm{d}\varphi^2$ derived in exercise (16.2). Think carefully what are M and its boundary ∂M (it is useful to introduce an asymptotic cut-off on the radial coordinate for calculations). For the bulk action use the fact that on-shell the Ricci-scalar is constant, $R = -6$. For the boundary recall (or recalculate) that extrinsic curvature is constant, $K = 2$. The divergent terms between bulk and boundary contributions must cancel exactly (if you are missing a factor 2 check again!), and you should get the result $\Gamma = -\frac{P_\tau P_\varphi}{16\pi}$, where $P_{\tau,\varphi}$ are the periodicities of the respective coordinates. Note that the periodicity of τ is 2π, but the periodicity of φ is $2\pi r_+$ in these coordinates, where r_+ obeys the following relations (for non-rotating BTZ with mass M and angular momentum J):

$$T = \frac{r_+}{2\pi} \qquad L = \bar{L} = \frac{r_+^2}{16} \qquad M = \frac{r_+^2}{8} \qquad J = 0$$

  Here T is the Hawking temperature [you have derived all these results implicitly when performing exercise (16.2), but if you have not done this exercise you are allowed to use the results above]. Your final result for the on-shell action should be $\Gamma = -\pi r_+/4$. The Helmholtz free energy multiplies this result with temperature, $F = T\Gamma$. Now it is easy to express F in terms of T. [Hints for the bonus part: $T = (r_+^2 - r_-^2)/(2\pi r_+)$, $L = (r_+ + r_-)^2/16$, $\bar{L} = (r_+ - r_-)^2/16$, $M = (r_+^2 + r_-^2)/8$, $J = r_+ r_-/4$; the periodicity of φ you have to figure out yourself...][1]

- This exercise is really short and simple, so try to do it without hints first. If you need hints use the online version and highlight this paragraph with your mouse — you should see the hidden text.

$$S = -\partial F/\partial T\big|_{\Omega=\text{const.}}.$$

$$S = A/4$$

  [Note: You can express entropy also as

$$S = 2\pi\sqrt{\frac{c_R L}{6}} + 2\pi\sqrt{\frac{c_L \bar{L}}{6}}$$

  which is nothing but the Cardy formula, providing a microscopic counting of the black hole entropy from a CFT perspective.]

- Think. Perhaps compare with exercise (8.3). Think again.

[1] A simple way to get the periodicity is to calculate the volume 2-form at the boundary in the "standard" t, ϕ coordinates, $\int \mathrm{d}^2x\sqrt{\gamma}|_{\hat{\rho}_c\gg 1} = 2\pi\beta e^{2\hat{\rho}_c}$, where $\beta = T^{-1}$ is the period of Euclidean time, and equate it to the volume 2-form in the simpler coordinates τ, φ, viz. $\int \mathrm{d}^2x\sqrt{\gamma}|_{\rho_c\gg 1} = 2\pi P_\varphi e^{2\rho_c}/4$. If you remember from exercise (16.2) that $e^{2\hat{\rho}_c} = 4\sqrt{L\bar{L}}e^{2\rho_c}$ you are nearly there.