Daniel Grumiller May 20th 2014

Black Holes II — Exercise sheet 9

(19.1) **Constraints in mechanics**
Derive all the constraints (primary, secondary, ...) in the Hamiltonian formulation for the Lagrangian

$$L(x,\,\dot{x},\,y,\,\dot{y},\,z,\,\dot{z}) = (\dot{z}-x)(\dot{y}-x)$$

and check if they are first or second class constraints.

(19.2) **Asymptotic symmetry algebra in Einstein gravity**
In exercise (17.2) some of you derived the transformation behavior of the state-dependent function $L(x^+)$ under boundary condition preserving gauge transformations of exercise (17.1) and found

$$\delta_{\epsilon^+}L(x^+) = \epsilon^+\partial_+L + 2L\partial_+\epsilon^+ - \tfrac{c}{12}\,\partial_+^3\epsilon^+$$

with $c = \frac{3\ell}{2G_N}$ (where ℓ is the AdS radius and G_N Newton's constant, both of which were set to unity in those exercises). By using the result for the general variation of the (say, left-moving) canonical boundary charges

$$\delta Q[\epsilon^+] \sim \int \mathrm{d}x^+\,\epsilon^+(x^+)\delta L(x^+)\,,$$

the relation to Poisson brackets

$$\delta_{\epsilon_1^+}Q[\epsilon_2^+] \sim \{Q[\epsilon_1^+],\,Q[\epsilon_2^+]\}\,,$$

and the Fourier decomposition

$$L(x^+) \propto \sum_{n\in\mathbb{Z}} e^{-inx^+}\,L_n$$

derive the asymptotic symmetry algebra generated by the L_n.

(19.3) **Central charge in Virasoro algebra**
Consider the vector fields $L_n = -z^{n+1}\partial_z$, which generate the Witt algebra

$$[L_n,\,L_m] = (n-m)\,L_{n+m}$$

Show that the central extension to the Virasoro algebra ($c,\,\tilde{c}\in\mathbb{R}$)

$$[L_n,\,L_m] = (n-m)\,L_{n+m} + \frac{c}{12}\,n(n^2-\tilde{c})\,\delta_{n+m,\,0}$$

is compatible with the Jacobi identities $[[L_n,\,L_m],\,L_k]+\mathrm{cycl}(n,m,k) = 0$. Moreover, show that we can always achieve $\tilde{c}=1$ by an appropriate shift of the generator $L_0 \to L_0+\ell_0$, where ℓ_0 is some c-number. Show finally that only for this choice the sub-algebra generated by L_{-1}, L_0, L_1 has no central extension. [You get bonus points if you show that the central extension above is unique.]

These exercises are due on May 27st 2014.

Hints:

- Compute first the primary constraints. You should get only one, namely the condition that one of the momenta is zero. Then construct the canonical Hamiltonian by the usual Legendre transformation, and add this constraint multiplied by some Lagrange multiplier. Check if the Poisson bracket of the primary constraint with the Hamiltonian vanishes. If not you get a secondary constraint. Repeat the procedure until no further constraints are obtained. Finally, check which constraints have (weakly) vanishing Poisson brackets among themselves and with the Hamiltonian — these are then first class constraints, while all other constraints are second class. How many physical degrees of freedom are in this system?

- I believe this might be a difficult exercise without further hints. What you are after is an expression of the form

$$[L_n,\, L_m] = f_{nm}{}^k L_k + C(n,\, m)\, \mathbb{1}$$

with structure constants f_{nm}^k and central term $C(n,\, m)$. To obtain commutators from Poisson brackets use the standard correspondence rule $i\{A,\, B\} \to [\hat{A},\, \hat{B}]$. In the current context this correspondence relation reads

$$i\delta_n Q_m \to [L_n,\, L_m]$$

where $\delta_n = \delta_{\epsilon_1^+}$ for the Fourier component $\epsilon_1^+ = e^{+inx^+}$ and $Q_m = Q(\epsilon_2^+)$ for the Fourier component $\epsilon_2^+ = e^{+imx^+}$. Please convince yourself that the statements in the hint so far make sense. If they do (actually, even if they do not...), then all that is left for you to do is to insert all the results given in the exercise into the correspondence relation above, using orthogonality and completeness results of Fourier analysis. This procedure then yields results for the structure constants $f_{nm}{}^k$ and the central term $C(n,\, m)$. You could compare your final result for the asymptotic symmetry algebra with the commutator algebra given in exercise (19.3).

- Just follow the instructions of the exercise: check first all Jacobi identities and then perform the shift of L_0 to achieve $\tilde{c} = 1$. It is straightforward to check for which generators the Virasoro algebra has vanishing central term. [The bonus part is less trivial.]