Daniel Grumiller June 8th 2020

Black Holes II — Exercise sheet 9

(19.1) **Hamilton–Jacobi recap**
Recall the Hamilton–Jacobi formulation of mechanics by deriving the Hamilton–Jacobi equation for Hamilton's principal function $S(q,t)$ for a given Hamiltonian $H(q,\,p)$

$$H\Big(q,\,\frac{\partial S}{\partial q}\Big)+\frac{\partial S}{\partial t}=0$$

You can start either with the Hamilton formulation or the Lagrange formulation or the Newton formulation of mechanics.

(19.2) **Extrinsic curvature practice**
Show that in Gaussian normal coordinates

$$\mathrm{d}s^2=\mathrm{d}\rho^2+h_{ij}(\rho,\,x^k)\,\mathrm{d}x^i\,\mathrm{d}x^j$$

the extrinsic curvature tensor $K_{\mu\nu}$ for some $\rho=$ const. hypersurface is given by

$$K_{ij}=\frac{1}{2}\,\partial_\rho h_{ij}\qquad\qquad K_{i\rho}=K_{\rho i}=K_{\rho\rho}=0\,.$$

Calculate the trace of extrinsic curvature for an infinitely long cylinder of radius ρ_c.

(19.3) **Variational étude**
Given a hypersurface with normal vector n^μ (with $n^\mu n_\mu=1$) calculate the variation of its covariant divergence, $\delta(\nabla_\mu n^\mu)$, up to total boundary derivative terms $\mathcal{D}_\mu(\dots)$, where $\mathcal{D}_\mu:=h^\nu_\mu\nabla_\nu$ and all free indices on which $\mathcal{D}_\mu$ acts must be projected to the boundary with $h_{\alpha\beta}=g_{\alpha\beta}-n_\alpha n_\beta$. Express the result entirely in terms of variations $\delta g_{\mu\nu}$.

These exercises are due on June 16th 2020.

Hints:

- Check any book on theoretical mechanics if you need a reminder. It is sufficient for this exercise to derive the Hamilton–Jacobi equation for the simplest case possible.

- Use the definitions in section 10.1, in particular $h_{\mu\nu} = g_{\mu\nu} - n_\mu n_\nu$ and $K_{\mu\nu} = h_\mu^\alpha h_\nu^\beta \nabla_\alpha n_\beta$, and split your coordinates into ρ and x^i. For the second part exploit cylindrical coordinates

$$\mathrm{d}s^2 = \mathrm{d}\rho^2 + \rho^2\, \mathrm{d}\varphi^2 + \mathrm{d}z^2 \qquad\qquad \varphi \sim \varphi + 2\pi$$

  and notice that they are Gaussian coordinates with respect to ρ. The boundary of the cylinder is the hypersurface $\rho = \rho_c$.

- Show first that the variation of such a normal vector (with lower index!) must be proportional to itself, $\delta n_\mu = \alpha n_\mu$, and vary then the equation $n^\mu n_\mu = 1$ to establish the factor α in terms of n^μ and $\delta g_{\mu\nu}$. Finally, make sure you do not forget that the variation acts also on the covariant derivative, in particular on the Christoffel connection therein. Use the result (4) of section 4.1 in the Black Holes II lecture notes (replacing h by δg) and try to cancel as many terms as possible in your final result. If you have more than two terms you could do better...
  Note: I am not sure what is the most efficient way of doing this calculation; I started with the variation $\delta(g^{\mu\nu}\nabla_\mu n_\nu)$ and proceeded from there, but there might be a shorter way. I often converted an expression like $h_\mu^\nu \nabla_\nu \delta g_{\alpha\beta} = h_\mu^\nu \nabla_\nu[(h_\alpha^\sigma + n_\alpha n^\sigma)(h_\beta^\tau + n_\beta n^\sigma)\delta g_{\sigma\tau}]$ to be able to use the boundary covariant derivative $\mathcal{D}_\mu$ acting on projected quantities (and then used the Leibnitz rule). Also, I used repeatedly identities like $n^\mu \nabla_\alpha n_\mu = 0$ and definitions like $h^{\mu\nu}\nabla_\mu n_\mu = K$ to bring the final result into the form (12) of section 10.2 in the lecture notes. While at intermediate steps you may encounter terms like $K^{\mu\nu}\delta g_{\mu\nu}$ or expressions containing $n^\mu \nabla_\mu n_\nu$, in the end all such terms cancel.