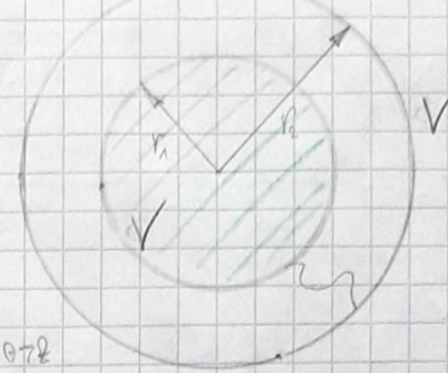
1 Problem Set, 12.10.2020, 14^{00}

1.1. If Photon Had Mass



$\vec{E} = -\vec{\nabla}\Phi \qquad \Delta\Phi = -\rho/\varepsilon_0 + \varphi/L^2$

a) Space between the shells $\rho = 0$

$\longrightarrow \Delta\Phi = \varphi/L^2 \quad , \quad \varphi(r) = u(r)/r \longrightarrow$ Ansatz

$$\Delta\Phi = \frac{1}{r^2}\partial_r\left(r^2\partial_r\varphi(r)\right) = \frac{\varphi(r)}{L^2} \qquad 0 \le r_1 < r < r_2$$

spherical symmetry

$$\frac{1}{r^2}\partial_r\left[r^2\left(\left(\partial_r u(r)\right)\frac{1}{r} - u(r)\frac{1}{r^2}\right)\right] = \frac{u(r)}{L^2 r}$$

$$\frac{1}{r^{\cancel{2}}}\partial_r\left[r\,\partial_r u(r) - u(r)\right] = \frac{u(r)}{L^2\cancel{r}}$$

$$\frac{1}{r}\left[\cancel{\partial_r u(r)} + r\,\partial_r^2 u(r) - \cancel{\partial_r u(r)}\right] = u(r)/L^2$$

$$\partial_r^2 u(r) = u(r)/L^2$$

$$u(r) = e^{\alpha r} \longrightarrow \alpha^2 u(r) = u(r)/L^2$$

$$\alpha^2 = 1/L^2 \longrightarrow \alpha = \pm 1/L$$

$$\varphi(r) = \frac{1}{r}\left[A e^{r/L} + B e^{-r/L}\right]$$

$$V = \frac{1}{r_1}\left[A e^{r_1/L} + B e^{-r_1/L}\right] \longrightarrow r_1 V = A e^{r_1/L} + B e^{-r_1/L}$$

$$B e^{-r_1/L} = r_1 V - A e^{r_1/L} \longrightarrow B = r_1 V e^{r_1/L} - A e^{2r_1/L}$$

$$r_2 V = A e^{r_2/L} + B e^{-r_2/L} = A e^{r_2/L} + e^{-r_2/L}\left(r_1 V e^{r_1/L} - A e^{2r_1/L}\right)$$

$$r_2 V - r_1 V e^{r_1/L} e^{-r_2/L} = A\left(e^{r_2/L} - e^{-r_2/L} e^{2r_1/L}\right)$$

$$A = \frac{r_2 V - r_1 V e^{r_1/L} e^{-r_2/L}}{e^{r_2/L} - e^{-r_2/L} e^{2r_1/L}} \longrightarrow B = r_1 V e^{r_1/L} - e^{2r_1/L} \cdot$$

$$\cdot \frac{r_2 V - r_1 V e^{r_1/L} e^{-r_2/L}}{e^{r_2/L} - e^{-r_2/L} e^{2r_1/L}}$$

$$\varphi(r) = V\frac{1}{r}\left[e^{r/L}\,\frac{r_2V - r_1V e^{r_1/L}e^{-r_2/L}}{e^{r_2/L} - e^{-r_2/L}e^{2r_1/L}} + e^{-r/L}\left(r_1 V e^{r_1/L} - e^{2r_1/L} \times \right.\right.$$

$$\left.\left. \times \frac{r_2V - r_1V\, e^{r_1/L}e^{-r_2/L}}{e^{r_2/L} - e^{-r_2/L}e^{2r_1/L}} \right)\right] =$$

$$= \frac{V}{r}\left[\frac{r_2 e^{r/L} - r_1 e^{r_1/L}e^{-r_2/L}e^{r/L}}{\text{Nenner}} + \frac{e^{-r/L}\, r_1 e^{r_1/L}\left(e^{r_2/L} - e^{-r_2/L}e^{2r_1/L}\right) - e^{-r/L}e^{2r_1/L} \times (r_2 - r_1 e^{r_1/L}e^{-r_2/L})}{\text{Nenner}} \right]$$

$$= \frac{V}{r}\left[r_2 e^{r/L} - r_1 e^{\frac{r_1-r_2+r}{L}} + r_1 e^{\frac{r_1+r_2-r}{L}} - \cancel{r_1 e^{\frac{3r_1-r_2-r}{L}}} - r_2 e^{\frac{2r_1-r}{L}} + \cancel{r_1 e^{\frac{3r_1-r_2-r}{L}}} \right]\frac{1}{\text{Nenner}} =$$

$$= \frac{V}{r}\left[r_2\,\frac{e^{r/L} - e^{2r_1/L}e^{-r/L}}{e^{r_2/L} - e^{-r_2/L}e^{2r_1/L}} + r_1\,\frac{e^{\frac{r_1+r_2-r}{L}} - e^{\frac{r_1-r_2+r}{L}}}{e^{r_2/L} - e^{-r_2/L}e^{2r_1/L}} \right] =$$

$$= V\left[\frac{r_2}{r}\,\frac{e^{\frac{r-r_1}{L}} - e^{\frac{r_1-r}{L}}}{e^{\frac{r_2-r_1}{L}} - e^{\frac{r_1-r_2}{L}}} + \frac{r_1}{r}\,\frac{e^{\frac{r_2-r}{L}} - e^{\frac{r-r_2}{L}}}{e^{\frac{r_2-r_1}{L}} - e^{\frac{r_1-r_2}{L}}} \right] =$$

$$= V\left[\frac{r_2}{r}\,\frac{\sinh\left((r-r_1)/L\right)}{\sinh\left((r_2-r_1)/L\right)} + \frac{r_1}{r}\,\frac{\sinh\left((r_2-r)/L\right)}{\sinh\left((r_2-r_1)/L\right)} \right]$$

$$\alpha := \frac{r_2-r_1}{L} \longrightarrow \varphi(r) = V\left[\frac{r_2}{r}\,\frac{\sinh\left(\frac{r-r_1}{L}\right)}{\sinh\alpha} + \frac{r_1}{r}\,\frac{\sinh\left(\frac{r_2-r}{L}\right)}{\sinh\alpha} \right]$$

$$\vec{E} = -\vec{\nabla}\varphi(r) = -\hat{r}\,\partial_r\varphi(r) =$$

$$= \frac{V\hat{r}}{\sinh\alpha}\left\{ r_2\left[\frac{\sinh\left(\frac{r-r_1}{L}\right)}{r^2} - \frac{\cosh\left(\frac{r-r_1}{L}\right)}{rL} \right] + r_1\left[\frac{\sinh\left(\frac{r_2-r}{L}\right)}{r^2} + \frac{\cosh\left(\frac{r_2-r}{L}\right)}{rL} \right] \right\}$$

b) $$-\nabla^2\varphi = \vec{\nabla}\cdot\vec{E} = \frac{\rho}{\varepsilon_0} - \frac{\varphi}{L^2}$$

$$\int_V \vec{\nabla}\cdot\vec{E}\,dV = \int_{\partial V} d\vec{S}\cdot\vec{E} = \int_V dV\left(\frac{\rho}{\varepsilon_0} - \frac{\varphi}{L^2}\right) = \frac{Q_{enc}}{\varepsilon_0} - \frac{1}{L^2}\int_V dV\,\varphi(\vec{r})$$

$$\oint_{\partial V_1} d\vec{S}\cdot\vec{E} = E(r_1)\cdot 4\pi r_1^2 = \frac{Q_1}{\varepsilon_0} - \frac{1}{L^2}\int_{V_1} dV\, \varphi(r)$$

$$4\pi r_1^2 \frac{V}{\sinh\alpha}\left\{ r_2\left[-\frac{1}{r_1 L}\right] + \cancel{r_1}\left[\frac{\sinh\left(\frac{r_2-r_1}{L}\right)}{r_1^{\cancel{2}}} + \frac{\cosh\left(\frac{r_2-r_1}{L}\right)}{\cancel{r_1}L}\right]\right\} =$$

$$= \frac{Q_1}{\varepsilon_0} - \frac{1}{L^2} V \frac{4\pi r_1^3}{3}$$

$$\frac{4\pi r_1^2 V}{\sinh\alpha}\left\{-\frac{r_2}{r_1 L} + \frac{\sinh\alpha}{r_1} + \frac{\cosh\alpha}{L}\right\} = \frac{Q_1}{\varepsilon_0} - \frac{V 4\pi r_1^3}{3L^2}$$

$$4\pi r_1^2 V\left\{\frac{1}{r_1} - \frac{r_2\,\mathrm{csch}\,\alpha}{r_1 L} + \frac{\coth\alpha}{L}\right\} = \frac{Q_1}{\varepsilon_0} - \frac{V 4\pi r_1^3}{3L^2}$$

$$Q_1 = 4\pi\varepsilon_0 r_1^2 V\left\{\frac{1}{r_1} + \frac{\coth\alpha}{L} - \frac{r_2}{r_1 L}\,\mathrm{csch}\,\alpha\right\} + \frac{4\pi\varepsilon_0}{3L^2} r_1^3 V$$

c) $L \to \infty \Rightarrow \alpha \to 0$

$\mathrm{csch}\, x \approx 1/x - x/6 \quad , \quad \coth x \approx 1/x + x/3$

$$\Rightarrow Q_1 \approx 4\pi\varepsilon_0 r_1^2 V \frac{(r_2-r_1)(2r_1+r_2)}{6L^2 r_1} + \frac{4\pi\varepsilon_0}{3L^2} r_1^3 V =$$

$$= \frac{2\pi\varepsilon_0}{3}\frac{r_1 V}{L^2}\left(\frac{r_2}{\cancel{1}}\right)^2\left(1 + \frac{r_1}{r_2}\right)$$

1.2 Consequences of Gauge Choices

a) $$\vec{\nabla}\cdot\vec{A}(\vec{r}) = \frac{\mu_0}{4\pi}\int d^3r'\, \vec{j}(\vec{r}')\,\vec{\nabla}\frac{1}{|\vec{r}-\vec{r}'|} = \cancel{\ }$$

$$\vec{\nabla}\frac{1}{r} = -\frac{\vec{r}}{r^3} \Rightarrow \vec{\nabla}\frac{1}{|\vec{r}-\vec{r}'|} = -\frac{\vec{r}-\vec{r}'}{|\vec{r}-\vec{r}'|^3}, \quad \vec{\nabla}'\frac{1}{|\vec{r}-\vec{r}'|} = +\frac{\vec{r}-\vec{r}'}{|\vec{r}-\vec{r}'|^3}$$

$$\cancel{\ } = -\frac{\mu_0}{4\pi}\int d^3r'\,\vec{j}(\vec{r}')\,\vec{\nabla}'\frac{1}{|\vec{r}-\vec{r}'|} = \quad = 0 \text{ for steady currents}$$

$$= -\frac{\mu_0}{4\pi}\left[\int d^3r'\,\vec{\nabla}'\left(\frac{\vec{j}(\vec{r}')}{|\vec{r}-\vec{r}'|}\right) - \int d^3r'\,\frac{\vec{\nabla}'\cdot\vec{j}(\vec{r}')}{|\vec{r}-\vec{r}'|}\right] =$$

Kontinuitätsgl.: $\frac{\partial\rho}{\partial t} + \vec{\nabla}\cdot\vec{j} = 0$

$$- \frac{\mu_0}{4\pi} \oint d\vec{S}' \cdot \frac{\vec{j}(\vec{r}\,')}{|\vec{r}-\vec{r}\,'|} \longrightarrow$$ surface integral vanishes at infinity, where we presume $\vec{j}(\vec{r}) \to 0$

b) Biot-Savart: $\vec{\nabla} \cdot \vec{B} = 0$ & $\vec{\nabla} \times \vec{B} = \mu_0 \vec{j}$

$$\rightarrow \quad \vec{B}(\vec{r}) = \vec{\nabla} \times \frac{\mu_0}{4\pi} \int d^3r' \, \frac{\vec{j}(\vec{r}\,')}{|\vec{r}-\vec{r}\,'|}$$

(Helmholtz theorem)

Biot-Savart formula in conventional form

$$\vec{B}(\vec{r}) = \frac{\mu_0}{4\pi} \int d^3r' \, \frac{\vec{j}(\vec{r}\,') \times (\vec{r}-\vec{r}\,')}{|\vec{r}-\vec{r}\,'|^3}$$ valid when

$\vec{\nabla} \cdot \vec{B} = 0$ & $\vec{\nabla} \times \vec{B} = \mu_0 \vec{j}$ $\longrightarrow$ necessary for HT

Compare with proposed formula

$$\vec{A}(\vec{r}) = \frac{1}{4\pi} \int d^3r' \, \frac{\vec{B}(\vec{r}\,') \times (\vec{r}-\vec{r}\,')}{|\vec{r}-\vec{r}\,'|^3}$$

$$\rightarrow \quad \vec{\nabla} \cdot \vec{A} = 0 \quad \& \quad \vec{\nabla} \times \vec{A} = \vec{B}$$

$\rightarrow$ COULOMB GAUGE

1.3 Aharonov - Bohm Effect

a) Coulomb gauge: $\vec{\nabla}\cdot\vec{A} = 0$ & $\vec{\nabla}\times\vec{A} = \vec{B} = B\vec{e}_z$

Due to sym. $\vec{A} = A\vec{e}_\varphi$ & $A = A(r)$

magnetic flux: $\Phi_B = \int_S d\vec{S}\cdot\vec{B} = \int_S d\vec{S}\cdot\vec{\nabla}\times\vec{A} = \int_{\partial S} d\vec{l}\cdot\vec{A}$

$r > R$ $\quad \Phi_B = B R^2 \cancel{\pi} = 2\cancel{\pi} r\, A(r)$

$r < R$ $\quad \Phi_B = B r^{\cancel{2}} \cancel{\pi} = 2\cancel{\pi}\cancel{r}\, A(r)$

$$A(r) = \begin{cases} Br/2 & , \; r < R \\ \dfrac{BR^2}{2r} & , \; r \geq R \end{cases} \longrightarrow A = A(r)\,\vec{e}_\varphi$$

$$r = R \quad \begin{cases} BR/2 \\ BR^2/2R = \dfrac{BR}{2} \end{cases} \checkmark$$

$\vec{\nabla}\cdot\vec{A} = 0 \longrightarrow$ Coulomb gauge

b) $\vec{A}' = \vec{A} + \vec{\nabla}\chi = \vec{A} + \vec{\nabla}\left(-\dfrac{\Phi_B \varphi}{2\pi}\right)$, Φ_B = const for $r > R$

$\vec{\nabla} = \vec{e}_r \partial_r + 1/S\, \vec{e}_\varphi \partial_\varphi + \vec{e}_z \partial_z$

$$\vec{A}' = \vec{A} - \frac{1}{2\pi r}\Phi_B \vec{e}_\varphi = \vec{A} - \frac{BR^2\cancel{\pi}}{2\cancel{\pi} r}\vec{e}_\varphi$$

$$\vec{A}' = \begin{cases} \left(Br/2 - \dfrac{BR^2}{2r}\right)\vec{e}_\varphi & , \; r < R \\ \left(\dfrac{BR^2}{2r} - \dfrac{BR^2}{2r}\right)\vec{e}_\varphi = 0 & , \; r \geq R \end{cases}$$

c) $\vec{B}' = \vec{\nabla}\times\vec{A}' = \vec{\nabla}\times(\vec{A} + \vec{\nabla}\chi) =$ (for $r < R$)

$$= \vec{B} - \frac{\Phi_B}{2\pi}\vec{\nabla}\times\left(\frac{\vec{e}_\varphi}{r}\right)$$

for $r \neq 0$: $\vec{\nabla}\times\left(\dfrac{1}{r}\vec{e}_\varphi\right) = \dfrac{1}{r}\partial_r\left(r\cdot 1/r\right)\vec{e}_z = 0$

Case $r = 0$ $\quad \int_S d\vec{S}\cdot\left[\vec{\nabla}\times\left(\dfrac{\vec{e}_\varphi}{r}\right)\right] = \oint_{\partial S} d\vec{l}\cdot\dfrac{\vec{e}_\varphi}{r} = \int_0^{2\pi} d\varphi = \underline{\underline{2\pi}}$

$$\Rightarrow \quad \vec{\nabla} \times \left(\frac{\vec{e}_\varphi}{r} \right) = \frac{\delta(r)}{r} \vec{e}_z$$

$$\Rightarrow \quad \vec{B}' = \vec{B} - \frac{\Phi_0}{2\pi} \frac{\delta(r)}{r} \vec{e}_z$$