1 If the photon had mass

If the photon had a mass m, the electric field would remain $\boldsymbol{E} = -\boldsymbol{\nabla}\varphi$ but Poisson's equation would change to include a charactaristic length $L = \hbar/mc$:

$$\nabla^2\varphi = -\frac{\rho}{\epsilon_0} + \frac{\varphi}{L^2}.$$

Experimental searches for m use a geometry first employed by Cavendish where two concentric conducting shells (radii $r_1 < r_2$) are maintained at a common potential V to ground by an infinitesimally thin connecting wire. When $m = 0$, all excess charge resides on the outside of the outer shell; no charge accumulates on the inner shell.

(a) Use the substitution $\varphi(r) = u(r)/r$ to solve the generalized Poisson equation above in the space between the shells. Find also the electric field in this region.

(b) Use the generalization of Gauss' law implied by the modified Poisson equation to find the charge Q on the inner shell.

(c) Show that, to leading order when $L \to \infty$,

$$Q \approx \frac{2\pi\epsilon_0 V}{3} r_1 \left(\frac{r_2}{L}\right)^2 \left(1 + \frac{r_1}{r_2}\right).$$

2 Consequences of gauge choices

(a) Show by direct calculation that the Coulomb gauge condition $\boldsymbol{\nabla} \cdot \boldsymbol{A} = 0$ applies to

$$\boldsymbol{A}(\boldsymbol{r}) = \frac{\mu_0}{4\pi} \int \mathrm{d}^3 r' \, \frac{\boldsymbol{j}(\boldsymbol{r}')}{|\boldsymbol{r} - \boldsymbol{r}'|}.$$

(b) Find the choice of gauge where a valid representation of the vector potential is

$$\boldsymbol{A}(\boldsymbol{r}) = \frac{1}{4\pi} \int \mathrm{d}^3 r' \, \frac{\boldsymbol{B}(\boldsymbol{r}') \times (\boldsymbol{r} - \boldsymbol{r}')}{|\boldsymbol{r} - \boldsymbol{r}'|^3}.$$

3 On the Aharonov–Bohm effect

(a) Find the vector potential inside and outside a solenoid that generates a non-vanishing magnetic field $\boldsymbol{B} = B\hat{\boldsymbol{z}}$ only inside an infinite cylinder of radius R. Work in the Coulomb gauge.

(b) Despite a vanishing magnetic field outside the solenoid, there are physical effects observable outside the solenoid. The Aharonov–Bohm effect occurs because the magnetic flux $\Phi_B = \oint \mathrm{d}\boldsymbol{s} \cdot \boldsymbol{A}$ is non-zero along a closed path outside the solenoid, say, a circle of radius $r > R$ which lies perpendicular to the solenoid axis. Show that the gauge transformation $\boldsymbol{A}' = \boldsymbol{A} + \boldsymbol{\nabla}\chi$ with $\chi = -\Phi_B\phi/2\pi$ and where ϕ is the angle in cylindrical coordinates leads to identically zero vector potential outside the solenoid.

(c) The result in part (b) suggests that we could eliminate the Aharonov–Bohm effect by a gauge transformation. Show, however, that the new magnetic field corresponds to a different physical problem, where

$$\boldsymbol{B}' = \boldsymbol{B} - \hat{\boldsymbol{z}} \, \frac{\Phi_B}{2\pi\rho} \delta(\rho)$$

and where ρ denotes the radius in cylindrical cooridnates.