4 Toroidal and poloidal magnetic fields

It is true (but not obvious) that any vector field $\boldsymbol{V}(\boldsymbol{r})$ which satisfies $\boldsymbol{\nabla}\cdot\boldsymbol{V}(\boldsymbol{r})=0$ can be written *uniquely* in the form

$$\boldsymbol{V}(\boldsymbol{r})=\boldsymbol{T}(\boldsymbol{r})+\boldsymbol{P}(\boldsymbol{r})=\boldsymbol{L}\psi(\boldsymbol{r})+\boldsymbol{\nabla}\times\boldsymbol{L}\gamma(\boldsymbol{r}),$$

where $\boldsymbol{L}=-\mathrm{i}\,\boldsymbol{r}\times\boldsymbol{\nabla}$ is the angular momentum operator and $\psi(\boldsymbol{r})$ and $\gamma(\boldsymbol{r})$ are scalar fields. $\boldsymbol{T}(\boldsymbol{r})=\boldsymbol{L}\psi(\boldsymbol{r})$ is called a *toroidal field* and $\boldsymbol{P}(\boldsymbol{r})=\boldsymbol{\nabla}\times\boldsymbol{L}\gamma(\boldsymbol{r})$ is called a *poloidal field.* This decomposition is widely used in laboratory plasma physics.

(a) Confirm that $\boldsymbol{\nabla}\cdot\boldsymbol{V}(\boldsymbol{r})=0$.

(b) Show that a poloidal current density generates a toroidal magnetic field and vice versa.

(c) Suppose there is no current in a finite volume V. Show that $\nabla^2\boldsymbol{B}(\boldsymbol{r})=\boldsymbol{0}$ in V.

(d) Show that $\boldsymbol{A}(\boldsymbol{r})$ in the Coulomb gauge is purely toroidal in V when $\psi(\boldsymbol{r})$ and $\gamma(\boldsymbol{r})$ are chosen so that $\nabla^2\boldsymbol{B}(\boldsymbol{r})=\boldsymbol{0}$ in V.

5 The magnetic field of charge in uniform motion

Consider a charge distribution $\rho(\boldsymbol{r})$ in rigid, uniform motion with velocity $\boldsymbol{v}$.

(a) Show that the magnetic field produced by this system is $\boldsymbol{B}(\boldsymbol{r})=(\boldsymbol{v}/c^2)\times\boldsymbol{E}$, where $\boldsymbol{E}(\boldsymbol{r})$ is the electric field produced by $\rho(\boldsymbol{r})$ at rest.

(b) Use this result to find $\boldsymbol{B}(\boldsymbol{r})$ for an infinite line of current and an infinite sheet of current (both uniform) from the corresponding electrostatic problem.

6 The London equations for a Superconductor

In 1935, the brothers Fritz and Heinz London described superconductivity using a phenomenological constitutive equation where a length $\delta>0$ relates the current density to the vector potential in the Coulomb gauge:

$$\boldsymbol{j}=-\frac{1}{\mu_0\delta^2}\boldsymbol{A}$$

(a) Use the London constitutive equation to derive a differential equation for $\boldsymbol{B}(\boldsymbol{r})$.

(b) The London theory predicts that $\boldsymbol{B}$ is not strictly zero at every point inside a superconductor. To see this, consider a slab of superconductor which is infinite in the x- and y-directions and lies between $z=-d$ and $z=d$. Compute $\boldsymbol{B}(z)$ inside the superconductor when the slab is placed in a static and uniform magnetic field $\boldsymbol{B}_0=B_0\hat{\boldsymbol{x}}$. Provide a sketch of $\boldsymbol{B}(z)$ in the slab.

7 Continuous creation

An early competitor of the Big Bang theory postulates the "continuous creation" of charged matter at a (very small) constant rate R at every point in space. In such a theory, the continuity equation is replaced by

$$\boldsymbol{\nabla} \cdot \boldsymbol{j} + \frac{\partial \rho}{\partial t} = R.$$

(a) For this to be true, it is necessary to alter the source terms in the Maxwell equations. Show that it is sufficient to modify Gauss' law to

$$\boldsymbol{\nabla} \cdot \boldsymbol{E} = \rho/\epsilon_0 - \lambda\varphi$$

and the Ampère–Maxwell law to

$$\boldsymbol{\nabla} \times \boldsymbol{B} = \mu_0 \boldsymbol{j} + \frac{1}{c^2}\frac{\partial \boldsymbol{E}}{\partial t} - \lambda \boldsymbol{A}.$$

Here, λ is a constant and φ and $\boldsymbol{A}$ are the usual scalar and vector potentials. Is this theory gauge invariant?

(b) Confirm that the modified Maxwell equations permit spherically symmetric solutions of the form

$$\boldsymbol{A}(r,t) = \boldsymbol{r} f(r,t) \qquad \text{and} \qquad \varphi(\boldsymbol{r},t) = \varphi_0$$

where $f(r,t)$ is a scalar function and φ_0 is a constant, and derive the resulting differential equation for $f(r,t)$.

(c) Show that the only non-singular solution to the partial differential equation satisfied by $f(r,t)$ is a constant.

(d) Show that the velocity of the charge created by this theory, $\boldsymbol{v} = \boldsymbol{j}/\rho$, is a linear function of r. This agrees with Hubble's famous observations.

8 Poincaré gauge

(a) Confirm that $\varphi(\boldsymbol{r}) = -\boldsymbol{r} \cdot \boldsymbol{E}$ and $\boldsymbol{A} = -\frac{1}{2}\boldsymbol{r} \times \boldsymbol{B}$ are acceptable scalar and vector potentials, respectively, for a constant electric field $\boldsymbol{E}$ and a constant magnetic field $\boldsymbol{B}$.

(b) By direct computation of $\boldsymbol{B} = \boldsymbol{\nabla} \times \boldsymbol{A}$ and $\boldsymbol{E} = -\boldsymbol{\nabla}\varphi - \partial \boldsymbol{A}/\partial t$, prove that the generalizations of the formulae in part (a) to arbitrary time-dependent fields are

$$\varphi(\boldsymbol{r},t) = -\boldsymbol{r} \cdot \int_0^1 \mathrm{d}\lambda\, \boldsymbol{E}(\lambda \boldsymbol{r}, t) \qquad \boldsymbol{A}(\boldsymbol{r},t) = -\int_0^1 \mathrm{d}\lambda\, \lambda \boldsymbol{r} \times \boldsymbol{B}(\lambda \boldsymbol{r}, t).$$

Hint: Prove first that $\frac{\mathrm{d}}{\mathrm{d}\lambda}\boldsymbol{G}(\lambda\boldsymbol{r}) = \frac{1}{\lambda}(\boldsymbol{r} \cdot \boldsymbol{\nabla})\, \boldsymbol{G}(\lambda \boldsymbol{r})$ for any vector field $\boldsymbol{G}$.

9 Energy flow in a coaxial cable

A cable is made from two coaxial cylindrical shells. The outer shell has radius b and charge per unit length λ, and carries a longitudinal current I. The inner cylinder has radius $a < b$ and charge per unit length $-\lambda$, and carries the current I back in the opposite direction.

(a) Integrate the Poynting vector to find the rate at which energy flows through a cross section of the cable.

(b) Show that a resistor R connected between the cylinders dissipates the power calculated in (a).