10 No electromagnetic bullets

(a) Let $f(\xi)$ be an arbitrary scalar function of the scalar variable ξ. The function $f(z-ct)$ is a traveling-wave solution of the one-dimensional wave equation

$$\left[\frac{\partial^2}{\partial z^2} - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right] f(z-ct) = 0.$$

Show that there exist localized solutions, i.e, $f(\xi)$ can be zero outside a finite interval of ξ.

(b) Let $\psi(x, y, z-ct)$ be a solution of the *three-dimensional* wave equation. Prove that ψ *cannot* be localized in the x, y, and z directions simultaneously.

11 Superposition and wave intensity

Let $\boldsymbol{E} = \boldsymbol{E}_1 + \boldsymbol{E}_2$ be the electric field of the sum of two monochromatic plane waves propagating in the z-direction. One wave has frequency ω_1 and is elliptically polarized. The other wave has frequency ω_2 and is elliptically polarized in a different way than the first wave. The intensity I is the time average of the Poynting vector over a time T that is much larger than any characteristic time scale. Derive precise, quantitative conditions which relate the averaging time T to ω_1 and ω_2 so the wave intensities satisfy $I = I_1 + I_2$.

12 Photon spin for plane waves

(a) Show that the angular momentum of an electromagnetic field in empty space without sources can be written in the form, partially in index notation

$$\boldsymbol{L}_{\text{EM}} = \epsilon_0 \int \mathrm{d}^3 r\, \boldsymbol{r} \times (\boldsymbol{E} \times \boldsymbol{B}) = \epsilon_0 \int \mathrm{d}^3 r\, E_k (\boldsymbol{r} \times \boldsymbol{\nabla}) A_k + \epsilon_0 \int \mathrm{d}^3 r\, \boldsymbol{E} \times \boldsymbol{A} = \boldsymbol{L}_{\text{orbital}} + \boldsymbol{L}_{\text{spin}}.$$

Note any requirements that the fields must satisfy at infinity. The last term is assigned to $\boldsymbol{L}_{\text{spin}}$ because it is a contribution to the angular momentum that does not depend on the "lever arm" $\boldsymbol{r}$.

(b) Show that the proposed decomposition is not gauge invariant and therefore not physically meaningful.

(c) Despite the foregoing, work in the Coulomb gauge and apply these formulae to a circularly polarized plane wave with electric field

$$\boldsymbol{E}_{\pm} = E_0 \frac{\hat{\boldsymbol{x}} \pm \mathrm{i}\hat{\boldsymbol{y}}}{\sqrt{2}} \mathrm{e}^{\mathrm{i}(kz-\omega t)}.$$

Show that the time averages obey $\pm\omega\hat{\boldsymbol{z}}\cdot\langle \boldsymbol{L}_{\text{spin}}\rangle = \langle U_{\text{EM}}\rangle$, where $U_{\text{EM}} = \int \mathrm{d}^3 r\, \frac{1}{2}\epsilon_0 \big(\mathrm{Re}^2(\boldsymbol{E}) + c^2 \mathrm{Re}^2(\boldsymbol{B})\big)$. Interpret this formula writing $\langle U_{\text{EM}}\rangle = \hbar\omega$.

13 An evanescent wave in vacuum

The electric field of a wave propagating in vacuum is $\boldsymbol{E} = \hat{\boldsymbol{y}} E_0 \exp[\mathrm{i}(hz - \omega t) - \kappa x]$.

(a) How are the real parameters h, κ, and ω related to one another?

(b) Find the associated magnetic field $\boldsymbol{B}$.

(c) Under what conditions is the polarization of the magnetic field close to circular?

(d) Compute the time-averaged Poynting vector.

14 Linear momentum of a wave packet

Let the complex electric field of an electromagnetic wave packet be

$$\boldsymbol{E}(\boldsymbol{r}, t) = \frac{1}{(2\pi)^{3/2}} \int \mathrm{d}^3 k \, \boldsymbol{E}_\perp(\boldsymbol{k}) \exp[\mathrm{i}(\boldsymbol{k} \cdot \boldsymbol{r} - ckt)].$$

(a) Show that the total linear momentum of the wave packet satisfies

$$c \, \boldsymbol{P}_{\mathrm{EM}} = \frac{\epsilon_0}{2} \int \mathrm{d}^3 k \, \hat{\boldsymbol{k}} \, |\boldsymbol{E}_\perp(\boldsymbol{k})|^2.$$

(b) Produce an argument which shows that

$$U_{\mathrm{EM}} \geq c \, |\boldsymbol{P}_{\mathrm{EM}}|,$$

(c) When does equality hold in Part (b)?

15 Charged particle motion in a circularly polarized plane wave

A particle with charge q and mass m interacts with a circularly polarized plane wave in vacuum. The electric field of the wave is $\boldsymbol{E}(z, t) = 2 \operatorname{Re}\{(\hat{\boldsymbol{x}} + \mathrm{i}\hat{\boldsymbol{y}}) \, E_0 \, \exp[\mathrm{i}(kz - \omega t)]\}$.

(a) Let $v_\pm = v_x \pm \mathrm{i} v_y$ and $\Omega = 2qE_0/mc$. Show that the equations of motion for the components of the particle's velocity $\boldsymbol{v}$ can be written

$$\frac{\mathrm{d}v_z}{\mathrm{d}t} = \frac{\Omega}{2} \left\{ v_+ \mathrm{e}^{+\mathrm{i}(kz - \omega t)} + v_- \mathrm{e}^{-\mathrm{i}(kz - \omega t)} \right\}$$
$$\frac{\mathrm{d}v_\pm}{\mathrm{d}t} = \Omega \, (c - v_z) \mathrm{e}^{\mp \mathrm{i}(kz - \omega t)}.$$

(b) Let $l_\pm = v_\pm \mathrm{e}^{\pm \mathrm{i}(kz - \omega t)} \pm \mathrm{i} c \, \Omega / \omega$ and show that

$$\frac{\mathrm{d}v_z}{\mathrm{d}t} = \frac{\Omega}{2}(l_+ + l_-) = \mathrm{i} \, \frac{\Omega}{2\omega} \, \frac{\mathrm{d}}{\mathrm{d}t}(l_+ - l_-).$$

(c) Differentiate the equations in part (a) and establish that

$$\frac{\mathrm{d}^2 v_z}{\mathrm{d}t^2} + \left[\Omega^2 + \omega^2\right] v_z = \omega^2 K,$$

where K is a real constant. Use the initial conditions $\boldsymbol{v}(0) = \boldsymbol{0}$ and $v_z'(0) = 0$ to evaluate K and solve for $v_z(t)$. Describe the nature of the particle acceleration in the z-direction.