16 Fresnel's problem for a topological insulator

The optical properties of a remarkable class of materials called *topological insulators* (TI) are captured by constitutive relations which involve the fine structure constant $\alpha = (e^2/\hbar c)/(4\pi\epsilon_0)$. With $\alpha_0 = \alpha\sqrt{\epsilon_0/\mu_0}$, the relations are

$$\boldsymbol{D} = \epsilon\boldsymbol{E} - \alpha_0\boldsymbol{B} \qquad\qquad \boldsymbol{H} = \frac{\boldsymbol{B}}{\mu} + \alpha_0\boldsymbol{E}.$$

(a) Begin with the Maxwell equations in matter with no free charge or current. Show that a monochromatic plane wave of $(\boldsymbol{E}, \boldsymbol{B})$ is a solution of these equations for a TI and find the wave speed.

(b) A plane wave with linear polarization impinges at normal incidence on the flat surface of a TI. Show that the transmitted wave remains linearly polarized with its electric field rotated by an angle θ_{F}. This is called *Faraday rotation* of the plane of polarization.

(c) Show that the reflected wave remains linearly polarized with its electric field rotated by an angle θ_{K}. This is called *Kerr rotation* of the plane of polarization.

17 The Lorenz–Lorentz and Drude formulae

Let the dielectric function $\epsilon(\omega) = \epsilon_0 n^2(\omega)$ characterize a macroscopic sphere of matter composed of N electrons. If the wavelength of the incident field is large compared to the sphere radius a, it is legitimate to use a quasistatic approximation. This problem equates two expressions for the polarization $\boldsymbol{P}(t)$ to find $\epsilon(\omega)$.

(a) Solve a quasi-electrostatic boundary value problem to find $\boldsymbol{P}(t)$ of the homogeneously polarized sphere when it is exposed to an external electric field $\boldsymbol{E}_0\cos(\omega t)$.

(b) Let the sphere have polarization $\boldsymbol{P}(t) = \boldsymbol{P}_0\cos(\omega t)$. Assume that each electron within the sphere is a lossless classical harmonic oscillator, where an electron displaced by $\boldsymbol{r}$ from its initial position is subjected to a restoring acceleration $-\omega_0^2\boldsymbol{r}$. Sum the dipole moments from all the electrons and equate the resulting polarization to the result of part (a) to get the Lorenz–Lorentz formula,

$$3\,\frac{n^2(\omega) - 1}{n^2(\omega) + 2} = \frac{\omega_{\mathrm{p}}^2}{\omega_0^2 - \omega^2},$$

with the plasma frequency $\omega_{\mathrm{p}}^2 = n_0 e^2/m\epsilon_0$ and the number density $n_0 = 3N/4\pi a^3$.

(c) Show that this reduces to the Drude formula for high frequencies,

$$n^2(\omega) = 1 - \frac{\omega_{\mathrm{p}}^2}{\omega^2}.$$

18 Loss and gain media

Consider the Lorentz-type index of refraction

$$n^2(\omega) = 1 + \frac{f\omega_{\mathrm{p}}^2}{\omega_0^2 - \omega^2 - \mathrm{i}\omega\Gamma}$$

The damping constant $\Gamma > 0$ and f is called the *oscillator strength*. Assume $|f| \ll 1$.

(a) Produce an argument based on monochromatic plane wave propagation that $f > 0$ describes an absorbing medium (like a conventional dielectric) that extracts energy from the field while $f < 0$ describes a gain medium (like a population of inverted atoms in a laser cavity) that supplies energy to the field.

(b) A wave packet propagates a distance L_A through an absorbing medium with $f_\mathrm{A} > 0$ immediately after it propagates a distance L_G through a gain medium with $f_\mathrm{G} < 0$. Under what conditions does the packet emerge undistorted from the absorbing medium? Hint: Do not make a group velocity (or any other) approximation to the sum of monochromatic plane waves that constitutes the packet.

19 Lorentz-model sum rule

The Lorentz-model dielectric function $\epsilon(\omega)/\epsilon_0 = n^2(\omega)$ given in problem 18 with $f = 1$ satisfies the f-sum rule

$$\int_0^\infty \mathrm{d}\omega\, \omega\, \mathrm{Im}\, \epsilon(\omega) = \frac{\pi}{2}\epsilon_0\, \omega_\mathrm{p}^2.$$

Show this explicitly for the case when the damping constant Γ is small.

20 Parseval's relation

(a) $\Delta(x)$ is an acceptable representation of a delta function if $\Delta(0)$ diverges and if it "filters" any smooth test function $f(x)$: $\int_{-\infty}^{\infty} \mathrm{d}x\, f(x)\Delta(x) = f(0)$. Show that these properties are satisfied by

$$\Delta(x) = \frac{1}{\pi^2}\int_{-\infty}^{\infty} \frac{\mathrm{d}y}{y(y-x)}.$$

(b) Let the real and imaginary parts of $\chi(\omega) = \chi_1(\omega) + \mathrm{i}\chi_2(\omega)$ satisfy the Kramers–Kronig relations. Use $\Delta(x)$ in part (a) to prove Parseval's relation,

$$\int_{-\infty}^{\infty} \mathrm{d}\omega\, |\chi_1(\omega)|^2 = \int_{-\infty}^{\infty} \mathrm{d}\omega\, |\chi_2(\omega)|^2.$$

21 A paramagnetic microwave amplifier

Let a transverse electromagnetic wave $\boldsymbol{H} = \hat{\boldsymbol{x}} H_x \exp\left[\mathrm{i}(ky - \omega t)\right]$ propagate in a linear magnetic medium exposed to a static magnetic field $\boldsymbol{B} = B_z \hat{\boldsymbol{z}}$. If γ and τ are constants, experiment shows that the induced magnetization obeys

$$\frac{\mathrm{d}\boldsymbol{M}}{\mathrm{d}t} = \gamma(\boldsymbol{M} \times \boldsymbol{B}) - \frac{|\boldsymbol{M}|}{\tau}(\hat{\boldsymbol{M}} - \hat{\boldsymbol{z}}).$$

(a) The first term on the right describes precession of the magnetization vector. The second term on the right side accounts (phenomenologically) for loss mechanisms that drive the system toward equilibrium (where $\boldsymbol{M}$ is aligned with the external field). Confirm this claim by computing $\boldsymbol{M}(t)$ when $\gamma = 0$

(b) Let $\omega_0^2 = \gamma^2 \mu_0 B_z H_z$ and show that

$$\frac{\mathrm{d}^2 M_x}{\mathrm{d}t^2} + \frac{2}{\tau}\frac{\mathrm{d}M_x}{\mathrm{d}t} + \left(\omega_0^2 + \tau^{-2}\right) M_x = \omega_0^2 \frac{M_z}{H_z} H_x.$$

Use this information to find the real and imaginary parts of the complex magnetic permeability $\mu(\omega)$. Establish that the magnetic permeability of the system is

$$\mu(\omega) = \mu_0 \left(1 + \frac{\omega_0^2}{\omega_0^2 + \tau^{-2} - \omega(\omega + 2\mathrm{i}/\tau)} \frac{M_z}{H_z}\right).$$

(c) Derive a wave equation for this medium and relate the real and imaginary parts of k to the real and imaginary parts of μ and to ϵ (assumed real, positive, and constant). Prove that the amplitude of the $\boldsymbol{H}$-wave given above decreases (increases) as it propagates if M_z/H_z is positive (negative).

Remark: Given the result in (c), amplification of the wave occurs only if we supply energy to "pump" the system into the higher energy state with $\boldsymbol{M}$ anti-parallel to $\boldsymbol{H}$. This is the analog of producing a "population inversion" to initiate laser action in an active medium.