Prerequisites

Pollack&Stump, Chapters 12 and 14, or Zangwill, Chapters 19 and 22

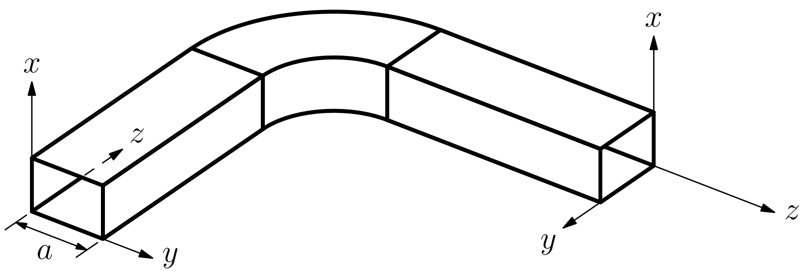
22 A vector-potential method

(a) Show that a general transverse magnetic (TM) wave in a hollow-tube waveguide can be derived from a longitudinal vector potential $\boldsymbol{A}(\boldsymbol{r},t) = \hat{\boldsymbol{z}}A(\boldsymbol{r}_\perp)\exp[\mathrm{i}(kz-\omega t)]$ which satisfies the wave equation.

(b) Duality implies that a general transverse electric (TE) wave can be derived from an "electric vector potential" $\tilde{\boldsymbol{A}}$, where $\boldsymbol{E}_{\mathrm{TE}} = \boldsymbol{\nabla}\times\tilde{\boldsymbol{A}}$. Explain why this makes perfectly good sense.

23 A waveguide with a bend

A rectangular waveguide with a constant cross section and perfectly conducting walls contains a curved section as sketched below. Also indicated is a local Cartesian coordinate system where the z-axis and y-axis remain tangent and normal to the walls, respectively.



(a) The scalar function Φ satisfies $[\nabla_\perp^2 + \omega^2/c^2]\Phi(y,z) = 0$, where $\nabla_\perp^2 = \partial^2/\partial x^2 + \partial^2/\partial y^2$. Note that Φ does not depend on x. Show that the four vacuum Maxwell equations and conducting-wall boundary conditions are satisfied by time-harmonic transverse electric (TE) modes of the form

$$\boldsymbol{E} = \hat{\boldsymbol{x}}\,\mathrm{i}\,\frac{\omega}{c}\Phi \qquad\qquad c\boldsymbol{B} = -\hat{\boldsymbol{x}}\times\boldsymbol{\nabla}\Phi.$$

(b) Suppose that the curvature $\kappa(z)$ of the side wall at any point on the guide satisfies $\kappa\, a \ll 1$ so the Laplacian operator in the local coordinate Cartesian coordinate system is well approximated by

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} + \frac{1}{2}\kappa^2(z).$$

Separate variables in the Helmholtz equation and show that propagating modes exist in the straight portion of the guide (at least) when $\omega > \pi c/a$.

(c) Show that at least one mode exists in the curved part of the guide for $\omega < \pi c/a$. Describe the spatial characteristics of this solution. Hint: Make an analogy with the one-dimensional, time-independent Schrödinger equation.

24 Resonant-frequency differences for a cavity (computer exercise)

A perfectly conducting resonant cavity has the shape of a rectangular box where the length, width, and height are chosen as three unequal irrational numbers. Evaluate (at least) the first 10^5 resonant frequencies numerically, label them so that $\omega_1 \leq \omega_2 \leq \ldots$, and construct a histogram of the nearest-neighbor frequency spacings of the form

$$P(s) = \sum_{k=1}^{N}\delta(\omega_{k+1}-\omega_k - s).$$

Show that your histogram is well approximated by a Poisson distribution. Hint: Be sure your histogram takes account of *all* frequencies less than a fixed maximum $\omega_{\max}$ (including degeneracies) and *none* greater than $\omega_{\max}$.

Use the tools of your preference (python, maxima, gnuplot, Fortran, Mathematica, ...). Please attach at least two extra files: the source file(s) in plain text and the histogram as a pdf document.

25 Lorentz transformation

(a) Derive the Lorentz transformation by assuming that the transformation is linear, and does not change the perpendicular coordinates. Write the transformation as

$$x' = A_1(x - vt), \qquad y' = y, \qquad z' = z, \qquad t' = A_2 t + A_3 x.$$

Determine A_1, A_2, A_3 by requiring that a flash of light produces an outgoing spherical wave, with velocity c, in either frame $\mathcal{F}$ or $\mathcal{F}'$.

(b) Let Δz and Δt be the difference between the space coordinates and the time coordinates of a pair of events. Show that, for at least some pairs of events, a Lorentz transformation of these differences *does not* reduce to a Galilean transformation in the limit of very low boost speed. Are the events in question space-like or time-like to each other?

26 The Lorentz group

All motion is along the x^1-axis and let $x^0 = ct$.

(a) Show that the Lorentz transformation from a frame $\mathcal{F}$ to a frame $\mathcal{F}'$ is a hyperbolic rotation about the hyperbolic angle $\theta = \operatorname{arctanh} v/c$, i.e.

$$\begin{aligned} x'^0 &= +x^0 \cosh\theta - x^1 \sinh\theta \\ x'^1 &= -x^0 \sinh\theta + x^1 \cosh\theta. \end{aligned}$$

The hyperbolic angle θ is called rapidity.

(b) Proof that all Lorentz transformations $\Lambda^\mu{}_\nu$ form a continuous group, the Lorentz group, by showing all four group axioms: identity, closure, associativity, and invertibility.

(c) Find the generator of the Lorentz group, which is the matrix $K^\mu{}_\nu$ such that all Lorentz transformations can be obtained from $\exp(\xi K)$ with $\xi \in \mathbb{R}$. How does ξ relate to θ? Hint: Work in the eigensystem of a general Lorentz transformation.

27 Linearity of the Lorentz transformation

It is sometimes stated that the linearity of the Lorentz transformation follows from the demand that uniform rectilinear motion in one frame $\mathcal{F}$ corresponds to uniform rectilinear motion in a different frame $\mathcal{F}'$. Show, to the contrary, that the same property is a consequence of the non-linear transformation law

$$x'^\mu = \frac{A^\mu{}_\nu x^\nu + b^\mu}{c_\nu x^\nu + d}$$

with the constants $A^\mu{}_\nu$, b^μ, c_ν, and d. Examine the points which transform to infinity and, using this information, invent a physical argument which forces $c_\nu = 0$. It is convenient to set $x^0 = ct$ so $v^0 = \mathrm{d}x^0/\mathrm{d}t = c$.