

2. Exercise of QFT for many-body systems

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4. Decay of an electron close to the Fermi sea

2+1=3 points

As it was discussed in the QFT Lecture when introducing the Fermi liquid theory, the possibility of treating a many-electron system in terms of independent excitations (i.e., the so-called “quasi-particles”) relies also on the fact that for metallic systems the electronic scattering between two particles at the Fermi Energy ε_F is vanishing. Specifically, as it was also mentioned in the Lecture, it is possible to show this by means of a phase-space argument: Consider the following elastic scattering process at $T = 0$: an incoming electron with fixed momentum \mathbf{k}_1 and energy $\varepsilon_1 > \varepsilon_F$ is scattered by one of the electrons of the metallic system (i.e., with momentum \mathbf{k}_2 and energy $\varepsilon_2 < \varepsilon_F$). If the scattering takes place, one will observe two outgoing particles with energies $\varepsilon_3, \varepsilon_4 > \varepsilon_F$. The corresponding “decay” probability for the incoming electron can be estimated as

$$\begin{aligned} \gamma &\propto \int d^d k_2 d^d k_3 d^d k_4 \delta(\mathbf{k}_1 + \mathbf{k}_2 - \mathbf{k}_3 - \mathbf{k}_4) \delta(\varepsilon_1 + \varepsilon_2 - \varepsilon_3 - \varepsilon_4) \\ &= \int d^d k_2 d^d k_3 \delta(\varepsilon_1 + \varepsilon_2 - \varepsilon_3 - \varepsilon_4), \end{aligned} \quad (1)$$

where the delta functions ensure the energy and momentum conservation of the process. The formally correct way of evaluating γ would be to perform the relatively cumbersome integrals over the momenta $\mathbf{k}_2, \mathbf{k}_3$ taking into account that ε_4 is a function of their lengths k_2, k_3 and of the angle between them.

a) A (much!) simpler way to evaluate the “decay” probability γ in $d > 2$, is to rewrite Eq. 1 as

$$\gamma \sim \int d\varepsilon_2 d\varepsilon_3 \quad (2)$$

and use the delta function only as a constraint to set the limits of the two energy integrals. Using Eq. 2 evaluate γ as a function of $\varepsilon_1, \varepsilon_F$ and interpret the results in terms of the Fermi-liquid theory.

b) How would Eq. 1 change in the case $T \neq 0$? And how would the estimate of **4a)** for γ change, assuming that for $T \neq 0$ the incoming electron with energy ε_1 is a thermally activated particle?

5. How to sum over Matsubara frequencies

2+1+2+2 = 5+2* points*

When performing the explicit evaluation of Feynman diagrams in terms of physical quantities, a typical intermediate step is the evaluation of sums over Matsubara frequencies. We will consider here the simplest cases, which represent, however, the basis for performing more complicated calculations occurring in realistic situations.

The particle density $\langle n \rangle$ of an electronic system can be expressed in term of the Green-function as follows

$$\langle n \rangle = \frac{1}{L^d} \sum_{\mathbf{k}} \langle c_{\mathbf{k}}^\dagger c_{\mathbf{k}} \rangle = \frac{1}{L^d} \sum_{\mathbf{k}} G(\mathbf{k}, \tau = 0^-) \quad (3)$$

where, as it was discussed in the Lecture, the fermionic Green's function in imaginary time is

$$G(\mathbf{k}, \tau) = -\frac{1}{Z} \text{Tr} \left\{ e^{-\beta \mathcal{H}} \mathbb{T} \left[c_{\mathbf{k}}(\tau) c_{\mathbf{k}}^\dagger(0) \right] \right\} \quad (4)$$

being Z the partition function, β the inverse temperature and \mathbb{T} is the imaginary-time ordering operator. Since, $G(\mathbf{k}, \tau)$ and its Fourier transform $G(\mathbf{k}, i\omega_n)$ are related in the following way:

$$G(\mathbf{k}, \tau) = \frac{1}{\beta} \sum_n e^{-i\omega_n \tau} G(\mathbf{k}, i\omega_n) \quad (5)$$

where the sum is over the Matsubara frequencies $\omega_n = \frac{\pi}{\beta}(2n+1)$, from the definition (4) we get

$$\langle n \rangle = \frac{1}{L^d} \sum_{\mathbf{k}} \frac{1}{\beta} \sum_n e^{-i\omega_n 0^-} G(\mathbf{k}, i\omega_n). \quad (6)$$

a) Perform the Matsubara sum in (6) for the case of non-interacting electrons with energy dispersion $\epsilon_{\mathbf{k}}$, whose Green function is given by $G(\mathbf{k}, i\omega_n) = \frac{1}{i\omega_n - \epsilon_{\mathbf{k}}}$ [*Hint: since $i\omega_n$ are the simple poles of the Fermi distribution function in the complex plane with residue $-\beta^{-1}$, the Matsubara sum can be written as an integral over a contour enclosing all Matsubara frequencies. Next, exploiting the analytic properties of the integrand, it is convenient to further transform such contour into two disconnected contours extending in the whole complex plane.*]

b) Think about a possible numerical implementation of Eq. (6), e.g. suppose one knows the value of $G(\mathbf{k}, i\omega_n)$ for a finite set of frequencies (say from $-i\omega_M$ to $i\omega_M$). What would be wrong with a “straightforward” numerical evaluation of such expression (i.e., just summing up all values available)? Suggest possible tricks to correct the problems encountered and to get reliable numerical results.

c) Often one has to calculate so-called “bubble” diagrams of the form

$$\frac{1}{L^d} \sum_{\mathbf{k}} \frac{1}{\beta} \sum_n G(\mathbf{k}, i\omega_n) G(\mathbf{k} + \mathbf{q}, i\omega_n + i\Omega_m) \quad (7)$$

where Ω_m is an “external” bosonic Matsubara frequency given by $2m\pi/\beta$. Using the free particle case again, calculate the expression (7) analytically, and discuss explicitly the results for the two limiting cases (i) $\Omega_m = 0, \mathbf{q} \rightarrow 0$ (“static limit”), or (ii) $\mathbf{q} = 0, \Omega_m \rightarrow 0$ (“dynamic limit”). [*Hint: a convenient way of proceeding is to use a partial fraction decomposition.*]

d) Consider again the Matsubara summation of **5a**), but now for an **interacting** electronic system, whose Green function can be written as $G(\mathbf{k}, i\omega_n) = \frac{1}{i\omega_n - \epsilon_{\mathbf{k}} - \Sigma(\mathbf{k}, i\omega_n)}$, where $\Sigma(\mathbf{k}, i\omega_n)$ is the electronic self-energy. While the Matsubara summation of (6) cannot be performed any longer exactly, it is possible to express it in terms of the Fermi function $f(\omega)$ and of the spectral function $A(\mathbf{k}, \omega) = -\frac{1}{\pi} \text{Im} G_R(\mathbf{k}, \omega)$. Derive such expression for the Matsubara summation of (6), and verify that it reproduces the explicit results of **5a**) in the case of a non-interacting system (i.e., when $\Sigma(\mathbf{k}, i\omega_n) = 0$).

* Bonus points

Viel Spaß!