IFP
TU Wien

Dr. A. Toschi, Dr. G. Sangiovanni, Dipl.-Ing. G. Rohringer

3. Exercise of QFT for many-body systems

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Intro: The Fermi-Liquid (FL) theory by Lev Landau, postulating the existence of independent quasi-particles with the same charge and spin of the original electrons but with energy $\epsilon_{\mathbf{p}} \neq \frac{p^2}{2m}$, allows to solve the "puzzle" of the metallic physics, that is to understand why the qualitative behavior of several observables in metals resembles so closely that of a non-interacting (!) electron gas. Starting point is the consideration that the energy change δE due to adding/removal of quasiparticles ($\delta n_{\mathbf{p},\sigma}$) with momentum $\mathbf{p}$ and spin σ to/from the Fermi sphere is given by the following functional:

$$\delta E[\delta n_{\mathbf{p},\sigma}] = \sum_{\mathbf{p},\sigma} \tilde{\epsilon}_{\mathbf{p}}\, \delta n_{\mathbf{p},\sigma} + \frac{1}{2} \sum_{\mathbf{p},\mathbf{p}';\,\sigma,\sigma'} f_{\sigma,\sigma'}(\mathbf{p},\mathbf{p}')\, \delta n_{\mathbf{p},\sigma}\, \delta n_{\mathbf{p}',\sigma'} + \cdots \tag{1}$$

The coefficients of the first term of the sum ($\tilde{\epsilon}_{\mathbf{p}}$) , which represent the energies for creating an excitation with momentum $\mathbf{p}$ <u>without</u> considering the feedback effects of the other quasiparticles (second term of Eq. (1)), are usually expanded (in the isotropic case) as $\tilde{\epsilon}_{\mathbf{p}} \sim \tilde{\epsilon}_F + \tilde{v}_F(p - p_F) + \cdots$, whereas $\tilde{v}_F = |\frac{\partial \tilde{\epsilon}_{\mathbf{p}}}{\partial \mathbf{p}}_{|p=p_F}| \simeq \frac{p_F}{m^*}$, being m^* the (enhanced) effective mass[1]. Finally, it is important to note that the quasi-particle distribution function $n_{\mathbf{p}}$ has the same form as for non-interacting electrons, but in terms of the quasiparticle energy $\epsilon_{\mathbf{p}}$.

6. First steps in calculations of a Fermi Liquid *1+1=2 points*

a) Verify that the so-called quasiparticle density of states defined as $\tilde{N}(\epsilon) = \frac{1}{L^d} \sum_{\mathbf{p}} \delta(\epsilon - \tilde{\epsilon}_{\mathbf{p}})$ can be easily expressed for $\epsilon = \tilde{\epsilon}_F$ as $\frac{m^*}{m} N(\epsilon_F)$, where $N(\epsilon_F)$ is the DOS of the corresponding non-interacting system.

b) Derive from Eq. 1 the formal expression of the (full) quasi-particle energy, defined as the energy necessary to add an excitation of momentum $\mathbf{p}$ close to the Fermi level, that is $\epsilon_{\mathbf{p}} = \frac{\delta E}{\delta n_{\mathbf{p},\sigma}}$. Which is the physical meaning of the term correcting the value of $\tilde{\epsilon}_{\mathbf{p}}$? Are the values of $\epsilon_{\mathbf{p}}$ depending on temperature or chemical potential? Motivate your answer.

7. Compressibility of Fermi-gases and Fermi Liquids *1*+2 points*

a) The isothermal compressibility -which is also related to the sound velocity in a given medium- is defined commonly in terms of pressure and volume as $\kappa_T = -\frac{1}{V}\frac{\partial V}{\partial P}\big|_T$. By using standard thermodynamic relations, express the pressure P in terms of the chemical potential μ, and demonstrate that κ_T can be also written as

$$\kappa_T = \frac{1}{n^2}\frac{\partial n}{\partial \mu}\bigg|_T, \tag{2}$$

with n being the total electron density. (Hint: The easiest way -but not the only one- is probably to rewrite the derivative of $\frac{\partial P}{\partial V}$ as a combination of derivatives in μ and n ...)

[1] As explained in the Lecture the mass enhancement can be related microscopically to the momentum and frequency derivatives of the electronic self-energy at the Fermi level.

b) Starting from Eq. (2), compute the value of the compressibility of a Fermi Liquid, neglecting all feedback effects in the calculations (Hint: express the variation of the total density n of the system as a sum over all quasiparticle states...). How does the obtained result compare with the compressibility of the corresponding non-interacting system? Are the feedback effects really playing no role in the calculations of κ_T?

8. Fermionic specific heat

1 +1+2+1=4+1* points*

The specific heat can be calculated from the entropy S as

$$c_V = T \frac{\partial}{\partial T} \left(\frac{S}{V} \right) \Big|_{V,\mu} .$$

In the grand canonical ensemble the entropy can be obtained as the derivative with respect to the temperature T of $k_B T \log \mathcal{Z}$, where k_B is the Boltzmann constant, $\beta = (k_B T)^{-1}$ and $\mathcal{Z}$ is the partition function, i.e. $\mathrm{Tr}\,[\,\exp\{-\beta(\mathcal{H} - \mu\mathcal{N})\}]$.

a) Calculate the entropy S of a non-interacting gas of fermions for which the partition function can be easily calculated from

$$\mathcal{H} = \sum_{\mathbf{p}} \epsilon_{\mathbf{p}} c^\dagger_{\mathbf{p}} c_{\mathbf{p}} \qquad \text{and} \qquad \mathcal{N} = \sum_{\mathbf{p}} c^\dagger_{\mathbf{p}} c_{\mathbf{p}}$$

b) Show that S is also equal to $-k_B \sum_{\mathbf{p}} [n_{\mathbf{p}} \log n_{\mathbf{p}} + (1 - n_{\mathbf{p}}) \log(1 - n_{\mathbf{p}})]$, where $n_{\mathbf{p}}$ is the Fermi distribution function, and use it to derive an expression for the specific heat according to the initial formula.

c) Rewrite the obtained expression for c_V as an integral over the energy $\epsilon^0_{\mathbf{p}} = \frac{p^2}{2m}$, using the DOS as you learned in the first exercise of this class. Then, by performing the variable substitution $x = \beta(\epsilon^0 - \mu)$ calculate the leading term of an expansion of c_V for small T (the so-called "Sommerfeld expansion"). To get a compact result assume that the DOS is slowly varying around the Fermi energy and use the fact that the dimensionless integral $\int dx\, x^2\, \mathrm{e}^x/(\mathrm{e}^x + 1)^2$, when evaluated between $-\infty$ and $+\infty$, equals $\pi^2/3$.

d) The same expression for the entropy given in **b)** holds for a Fermi liquid if one replaces $n_{\mathbf{p}}$ with the quasi-particle distribution function $1/(\mathrm{e}^{(\beta(\epsilon_{\mathbf{p}} - \mu)} + 1)$, $\epsilon_{\mathbf{p}}$ being the quasi-particle energy (see Eq. 1, and **6b)**). Going through exactly the same steps to derive c_V for the non-interacting Fermi gas, one notices that a new term appears. Determine such term. It can be shown, however, that such term can be neglected for this calculation. Derive hence the final expression for the specific heat of a Fermi liquid and discuss how it compares to the non-interacting c_V.

* Bonus points

Viel Spaß!