IFP
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6. Übung QFT für Vielteilchen-Systeme

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Intro: The Fermi-Liquid (FL) theory by Lev Landau, postulating the existence of independent quasi-particles with the same charge and spin of the original electrons but with energy $\epsilon_{\mathbf{k}} \neq \frac{k^2}{2m}$ ($\hbar \equiv 1$), allows to solve the "puzzle" of the metallic physics, that is to understand why the qualitative behavior of several observables in metals resembles so closely that of a non-interacting (!) electron gas. Historical starting point of the Landau theory is the consideration that the energy change δE due to adding/removal of quasiparticles ($\delta n_{\mathbf{k},\sigma}$) with momentum $\mathbf{k}$ and spin σ to/from the Fermi sphere is given by the following functional:

$$\delta E[\delta n_{\mathbf{k},\sigma}] = \sum_{\mathbf{k},\sigma} \tilde{\epsilon}_{\mathbf{k}}\, \delta n_{\mathbf{k},\sigma} + \frac{1}{2} \sum_{\mathbf{k},\mathbf{k}';\,\sigma,\sigma'} f_{\sigma,\sigma'}(\mathbf{k},\mathbf{k}')\, \delta n_{\mathbf{k},\sigma}\, \delta n_{\mathbf{k}',\sigma'} + \cdots \tag{1}$$

The coefficients of the first term of the sum ($\tilde{\epsilon}_{\mathbf{k}}$) , which represents the energies for creating an excitation with momentum $\mathbf{k}$ <u>without</u> considering the feedback effect ($f_{\sigma,\sigma'}$) of the other quasiparticles, are usually expanded (in the isotropic case) as $\tilde{\epsilon}_{\mathbf{k}} \sim \tilde{\epsilon}_F + \tilde{v}_F(k-k_F) + \cdots$, whereas $v_F = |\frac{\partial \tilde{\epsilon}_{\mathbf{k}}}{\partial \mathbf{k}}_{|k=k_F}| \simeq \frac{k_F}{m^*}$, being m^* the (enhanced) effective mass . Finally, it is important to note that the quasi-particle distribution function $n_{\mathbf{k}}$ has the same form of as for non-interacting electrons, but in term of the quasiparticle energy $\epsilon_{\mathbf{k}}$.

1. First steps in calculations of a Fermi Liquid *2+1=3 Punkte*

a) Verify that the so-called quasiparticle density of states for $\epsilon = \tilde{\epsilon}_F$ defined as $\tilde{N}(\epsilon) = \frac{1}{L^d} \sum_{\mathbf{k}} \delta(\epsilon - \tilde{\epsilon}_{\mathbf{k}})$ can be easily expressed as $\frac{m^*}{m} N(\epsilon_F)$, where $N(\epsilon_F)$ is the DOS of the corresponding non-interacting system.

b) Derive from Eq. 1, the formal expression of the (full) quasi-particle energy, defined as the energy necessary to add an excitation of momentum $\mathbf{k}$ close to the Fermi level, that is $E_{\mathbf{k}} = \frac{\delta E}{\delta n_{\mathbf{k},\sigma}}$. Which is the physical meaning of the term correcting the value of $\tilde{\epsilon}_{\mathbf{k}}$? Are the values of $\epsilon_{\mathbf{k}}$ depending on temperature or chemical potential? Motivate your answer.

2. Specific heat of fermions

1+2+1=4 Punkte

The specific heat can be calculated from the entropy S as

$$c_V = T\frac{\partial}{\partial T}\left(\frac{S}{V}\right)\Big|_{V,\mu}.$$

In the grand canonical ensemble the entropy can be obtained as the derivative with respect to the temperature T of $k_B T \log \mathcal{Z}$, where k_B is the Boltzmann constant, β is $(k_B T)^{-1}$ and $\mathcal{Z}$ is the partition function, i.e. $\mathrm{Tr}\exp\{-\beta(\mathcal{H}-\mu\mathcal{N})\}$. For non-interacting electrons, this leads to

$$S = -k_B \sum_{\mathbf{k}} \left[n_{\mathbf{k}} \log n_{\mathbf{k}} + (1-n_{\mathbf{k}})\log(1-n_{\mathbf{k}})\right] \tag{2}$$

where $n_{\mathbf{k}}$ is the Fermi distribution function.

a) Starting from the explicit expression for the entropy (Eq. 2) derive an expression for c_V as an integral over the energy $\epsilon_{\mathbf{k}} = \frac{k^2}{2m}$, using the DOS as you learned in the 4$^{\text{th}}$ exercise of this class. Then, by performing the variable substitution $x = \beta(\epsilon^0 - \mu)$ calculate the leading term of an expansion of c_V for small T (the so-called "Sommerfeld expansion"). To get a compact result assume that the DOS is slowly varying around the Fermi energy and use the fact that the dimensionless integral $\int dx\, x^2\, \mathrm{e}^x/(\mathrm{e}^x+1)^2$, when evaluated between $-\infty$ and $+\infty$ equals $\pi^2/3$.

b) The same expression of Eq. 2 can be used for for a Fermi liquid[1] if one replaces $n_{\mathbf{k}}$ with the quasi-particle distribution function $1/(\exp(\beta(E_{\mathbf{k}}-\mu)+1)$, $E_{\mathbf{k}}$ being the quasi-particle energy (see Eq. 1, and **1b)**). Going through exactly the same steps to derive c_V for the non-interacting Fermi gas, one notices that a new term appears. Determine such term. It can be shown, however, that such term can be neglected for this calculation. Derive hence the final expression for the specific heat of a Fermi liquid and discuss how it compares to the non-interacting c_V.

3. Compressibility of Fermi-gases and Fermi Liquids

2 Punkte*

a) The isothermal compressibility -which is also related to the sound velocity in a given medium- is defined commonly in terms of pressure and volume as $\kappa_T = -\frac{1}{V}\frac{\partial V}{\partial P}\big|_T$. By using standard thermodynamic relations, one can express the pressure P in terms of the chemical potential μ, and hence, κ_T can be also written as

$$\kappa_T = \frac{1}{n^2}\frac{\partial n}{\partial \mu}\Big|_T, \tag{3}$$

being n the total electron density. Compute the compressibility of a Fermi Liquid. (Hint: Express the variation of total density n of the system as a sum over all quasiparticle states, neglecting all feedback effects). How does the obtained result compare with the compressibility of the corresponding non-interacting system? Are the feedback effects really playing no role in the calculations of κ_T?

* Bonus points

Viel Spaß!

[1] *more precisely for the low temperature changes of the entropy*