IFP
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7. Übung QFT für Vielteilchen-Systeme

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1. Second order Feynman diagrams *2+1=3 Punkte*

When considering the perturbation theory of the Green's function, at the second order, one has to calculate, using Wick's theorem, all the contractions/Feynman diagrams of

$$-\langle T_\tau c_{\mathbf{k}}(\tau) c^\dagger_{\mathbf{k}} H_V(\tau_1) H_V(\tau_2)\rangle, \tag{1}$$

where

$$H_V = \frac{1}{(2\pi)^6}\int d^3k'' \int d^3k' \int d^3q \;\, c^\dagger_{\mathbf{k}''+\mathbf{q}} c^\dagger_{\mathbf{k}'-\mathbf{q}} \frac{V(q)}{2} c_{\mathbf{k}'} c_{\mathbf{k}''}. \tag{2}$$

Due to the Linked Cluster Theorem only the connected contributions need to be considered.

a) Draw ten among all second order Feynman diagrams corresponding to ten connected second order contractions.

b) For one of these Feynman diagrams explicitly write the analytic espression in terms of Green's functions.

2. How to sum over Matsubara frequencies *2+1+2+2*=5+2* Punkte*

When performing the explicit evaluation of Feynman diagrams in terms of physical quantities, a typical intermediate step is the evaluation of sums over Matsubara frequencies. We will consider here the simplest cases, which represent, however, the basis for performing more complicate calculations occurring in realistic situations.

The particle density $\langle n\rangle$ of an electronic system can be expressed in term of the Green-function as follows

$$\langle n\rangle = \frac{1}{L^d}\sum_{\mathbf{k}}\langle c^\dagger_{\mathbf{k}} c_{\mathbf{k}}\rangle = \frac{1}{L^d}\sum_{\mathbf{k}} G(\mathbf{k}, \tau = 0^-) \tag{3}$$

where, as it was discussed in the Lecture, the fermionic Green's function in imaginary time is

$$G(\mathbf{k},\tau) = -\frac{1}{Z}\mathrm{Tr}\left\{\mathrm{e}^{-\beta\mathcal{H}}\mathrm{T}\left[c_{\mathbf{k}}(\tau)c^\dagger_{\mathbf{k}}(0)\right]\right\} \tag{4}$$

being Z the partition function, β the inverse temperature and T is the imaginary-time ordering operator. Since $G(\mathbf{k},\tau)$ and its Fourier transform $G(\mathbf{k}, i\omega_n)$ are related in the following way:

$$G(\mathbf{k},\tau) = \frac{1}{\beta}\sum_n \mathrm{e}^{-i\omega_n\tau} G(\mathbf{k}, i\omega_n) \tag{5}$$

where the sum is over the Matsubara frequencies $\omega_n = \frac{\pi}{\beta}(2n+1)$, from the definition (4) we get

$$\langle n\rangle = \frac{1}{L^d}\sum_{\mathbf{k}}\frac{1}{\beta}\sum_n \mathrm{e}^{-i\omega_n 0^-} G(\mathbf{k}, i\omega_n). \tag{6}$$

a) Perform the Matsubara sum in (6) for the case of non-interacting electrons with energy dispersion $\epsilon_{\mathbf{k}}$, whose Green function is given by $G(\mathbf{k}, i\omega_n) = \frac{1}{i\omega_n - \epsilon_{\mathbf{k}}}$ [*Hint: since* $i\omega_n$ *are the simple poles of the Fermi distribution function in the complex plane with residue* $-\beta^{-1}$*, the Matsubara sum can be rewritten as an integral over a contour enclosing all Matsubara frequencies. Next, exploiting the analytic properties of the integrand, it is convenient to further transform such contour into two disconnected contours extending in the whole complex plane...*]

b) Think about a possible numerical implementation of Eq. (6), e.g. suppose one knows the value of $G(\mathbf{k}, i\omega_n)$ for a finite set of frequencies (say from $-i\omega_M$ to $i\omega_M$) . What would be wrong with a "straightforward" numerical evaluation of such expression (i.e., just summing up all values available)? Suggest possible tricks to correct the problems encountered and to get reliable numerical results.

c) Often one has to calculate so-called "bubble" diagrams of the form

$$\frac{1}{L^d} \sum_{\mathbf{k}} \frac{1}{\beta} \sum_{n} G(\mathbf{k}, i\omega_n) G(\mathbf{k} + \mathbf{q}, i\omega_n + i\Omega_m) \tag{7}$$

where Ω_m is an "external" bosonic Matsubara frequency given by $2m\pi/\beta$. Using the free particle case again, calculate the expression (7) analytically, and discuss explicitly the results for the two limiting cases (i) $\Omega_m = 0, \mathbf{q} \to 0$ ("static limit"), or (ii) $\mathbf{q} = 0, \Omega_m \to 0$ ("dynamic limit"). [*Hint: a convenient way of proceeding is to use a partial fraction decomposition.*]

d) Consider again the Matsubara summation of **5a)**, but now for an **interacting** electronic system. Use the spectral representation of the Green function $G(\mathbf{k}, i\omega_n)$ to rewrite the Matsubara of (6) in terms of the Fermi function $f(\omega)$ and of the spectral function $A(\mathbf{k}, \omega) = -\frac{1}{\pi} \mathrm{Im} G_R(\mathbf{k}, \omega)$. Verify that it reproduces the explicit results of **5a)** in the case of a non-interacting system (i.e., when $G(\mathbf{k}, i\omega_n) = \frac{1}{i\omega_n - \epsilon_{\mathbf{k}}}$).

* Bonus points

Viel Spaß!