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3. Exercise on QFT for many-body systems

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Intro: The Fermi liquid (FL) theory by Lev Landau, postulating the existence of independent quasi-particles with the same charge and spin of the original electrons but with energy $\epsilon_{\mathbf{p}} \neq \frac{p^2}{2m}$, allows to solve the "puzzle" of the metallic physics, that is to understand why the qualitative behavior of several observables in metals resembles so closely that of a non-interacting (!) electron gas. Starting point of this phenomenological approach is the consideration that the energy change δE due to adding/removal of quasiparticles ($\delta n_{\mathbf{p},\sigma}$) with momentum $\mathbf{p}$ and spin σ to/from the Fermi sphere is given by the following functional (note that in the truly non-interacting case, this would just be a number) :

$$\delta E[\delta n_{\mathbf{p},\sigma}] = \sum_{\mathbf{p},\sigma} \tilde{\epsilon}_{\mathbf{p}}\, \delta n_{\mathbf{p},\sigma} + \frac{1}{2} \sum_{\mathbf{p},\mathbf{p}';\sigma,\sigma'} f_{\sigma,\sigma'}(\mathbf{p},\mathbf{p}')\, \delta n_{\mathbf{p},\sigma}\, \delta n_{\mathbf{p}',\sigma'} + \cdots \tag{1}$$

The coefficients of the first term of the sum ($\tilde{\epsilon}_{\mathbf{p}}$) , which represents the energies for creating an excitation with momentum $\mathbf{p}$ <u>without</u> considering the feedback effect of the other quasiparticles, are usually expanded (in the isotropic case) as $\tilde{\epsilon}_{\mathbf{p}} \sim \tilde{\epsilon}_F + \tilde{v}_F(p - p_F) + \cdots$, whereas $v_F = |\frac{\partial \tilde{\epsilon}_{\mathbf{p}}}{\partial \mathbf{p}}_{|p=p_F}| \simeq \frac{p_F}{m^*}$, being m^* the (enhanced) effective mass[1]. Finally, it is important to note that the quasi-particle distribution function $n_{\mathbf{p}}$ has the same form of as for non-interacting electrons, but in term of the quasiparticle energy $\epsilon_{\mathbf{p}}$.

5. First steps in calculating a Fermi liquid (FL) *1+1*+2=3+1* points*

a) Verify that the so-called quasiparticle density of states for $\epsilon = \tilde{\epsilon}_F$ defined as $\tilde{N}(\epsilon) = \frac{1}{L^d} \sum_{\mathbf{p}} \delta(\epsilon - \tilde{\epsilon}_{\mathbf{p}})$ can be easily expressed as $\frac{m^*}{m} N(\epsilon_F)$, where $N(\epsilon_F)$ is the DOS of the corresponding non-interacting system.

b) Derive from Eq. 1, the formal expression of the (full) quasi-particle energy, defined as the energy necessary to add an excitation of momentum $\mathbf{p}$ close to the Fermi level, that is $\epsilon_{\mathbf{p}} = \frac{\delta E}{\delta n_{\mathbf{p},\sigma}}$. Which is the physical meaning of the term correcting the value of $\tilde{\epsilon}_{\mathbf{p}}$? Are the values of $\epsilon_{\mathbf{p}}$ depending on temperature or chemical potential? Motivate your answer.

c) Calculate the specific heat at constant volume c_V for the non-interacting Fermi gas in three dimensions:

$$c_V = \left(\frac{\partial E}{\partial T}\right)_V$$

where for this specific case

$$E = E_{\text{kin}} = \left\langle \sum_{\vec{p},\sigma} \epsilon_p c^\dagger_{\vec{p},\sigma} c_{\vec{p},\sigma} \right\rangle \quad \text{with } \epsilon_p = \frac{p^2}{2m}$$

What is the temperature dependence of c_V? How will the final result change for interacting electrons under the assumption that the Fermi liquid theory can be applied?

[1] As explained in the Lecture the mass enhancement can be related microscopically to the momentum and frequency derivatives of the self-energy at the Fermi level (see also exercise 6).

6. FL parameters from a microscopic theory *1+2+1=4 points*

In interacting microscopic theories one of the most important quantities is, arguably, the self-energy $\Sigma(\vec{k}, \omega)$. In the lecture it was shown that for a metal one can connect values extracted out of a Taylor expansion of the self-energy around the Fermi level to the phenomenological parameters of the Landau Fermi liquid theory.

In particular, if the self-energy out of the microscopic theory considered (e.g. dynamical mean field theory) is purely local [i.e., not $\vec{k}$-dependent: $\Sigma(\vec{k}, \omega) = \Sigma(\omega)$], one obtains the following relations:

$$Z = \left[1 - \frac{\partial \text{Re } \Sigma(\omega)}{\partial \omega}\bigg|_{\omega \to 0}\right]^{-1}, \quad \frac{m^*}{m} = Z^{-1}, \quad \Gamma = -\text{Im } \Sigma(\omega)\big|_{\omega \to 0} \tag{2}$$

where Z is the quasiparticle weight, m^* the effective mass and Γ defines the quasiparticle scattering rate so that the quasiparticle lifetime is $\tau = (2Z\Gamma)^{-1}$.

Now consider the file `siw-data.txt`. It contains local fermionic self-energies on the positive Matsubara axis $\Sigma_c(i\nu_n)$ for four different physical cases $c = 1, 2, 3, 4$ at the inverse temperature $\beta = 50$ eV^{-1}. The format of the columns in the file is ν_n — Re $\Sigma_1(i\nu_n)$ — Im $\Sigma_1(i\nu_n)$ — Re $\Sigma_2(i\nu_n)$ ...

a) Plot the real and imaginary parts of the self-energies in all cases, respectively.

b) For cases one to three numerically extract the quasiparticle weight Z, the effective mass $\frac{m^*}{m}$ and Γ. What is the difference between these cases and how could one interpret those? *Hint: In principle one would have to analytically continue the self-energy from the Matsubara to the real frequency axis ($i\nu_n \to \omega + i\delta$). However, indicative results can also be obtained by simply using a polynomial fit of $\Sigma(i\nu_n)$ in the region of low (positive) Matsubara frequencies and extracting the parameters out of it thereafter via the following modified formulas:*

$$Z = \left[1 - \frac{\partial Im\ \Sigma(i\nu_n)}{\partial \nu_n}\bigg|_{\nu_n \to 0}\right]^{-1}, \quad \Gamma = -Im\ \Sigma(i\nu_n)\big|_{\nu_n \to 0}$$

c) What about the fourth case?

7. From causality to Kramers-Kronig relations *3 points*

In this exercise it should be demonstrated that the Kramers-Kronig relations for a frequency-dependent function $\tilde{f}(\omega)$ are a consequence of the *causality* of its inverse Fourier transform $f(t)$ in the time regime, i.e., of the property $f(t) \equiv 0$ for $t < 0$. The goal is to derive the Kramers-Kronig relations for $\tilde{f}(\omega)$ by starting from the identity $f(t) = f(t)\theta(t)$. *Hint: For your calculations use the convolution theorem and the Fourier transform of the θ-function, which has been discussed in the lecture.*

* Bonus points

Viel Spaß!