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4. Exercise on QFT for many-body systems

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7. Linked-cluster theorem *3 points*

As it has already been discussed in the Lecture, only connected Feynman diagrams have to be considered when calculating the one-particle Green function. Starting from the perturbation expansion of the Green's function at $T \neq 0$, show that the time-ordered average for a given order n of perturbation theory decomposes into a product of connected and a disconnected diagrams. Prove that the disconnected factor cancels exactly the denominator $Z = \langle S(\beta) \rangle_0$.

[*Hint: Consider the n-th order term in the perturbation expansion of the numerator of the Green function (*$\sim \langle T\, c(\tau)c^\dagger(0)H_V(\tau_1)\cdots H_V(\tau_n)\rangle_0$*). According to Wick's theorem this can be written in terms of connected (*$\sim \langle T\, c(\tau)c^\dagger(0)H_V(\tau_1)\cdots H_V(\tau_m)\rangle_0$*) and disconnected (*$\sim \langle T\, H_V(\tau_{m+1})\cdots H_V(\tau_n)\rangle_0$*) contractions, with* $m = 1,\cdots,n$.]

8. Application of the Wick Theorem: charge/density fluctuations in the non-interacting case *2+2+1+2+2*=7+2* points*

One of the most important building blocks of the physics encoded in the Feynman diagrammatics of many-electons systems is the description of the electronic (negatively charged) density-fluctuations.
In fact, as it will be discussed in the upcoming Lectures, the fluctuations of the electronic density around its mean value $\bar{n}$ are responsible for the significant screening of the (otherwise strong) Coulomb repulsion. These screening processes -ultimately- make possible the Fermi-liquid behavior of electrons in metallic systems.

The general expression for the density-fluctuations in imaginary time is given by:

$$\chi_{nn}(\mathbf{r}-\mathbf{r}', \tau-\tau') = \left\langle T_\tau\, \hat{n}(\mathbf{r},\tau)\hat{n}(\mathbf{r}',\tau')\right\rangle, \tag{1}$$

where the electron-density operator at the position $\mathbf{r}$ and (imaginary) time τ is defined as $\hat{n}(\mathbf{r},\tau) = \sum_{\sigma=\uparrow,\downarrow} \hat{\psi}^\dagger_\sigma(\mathbf{r},\tau)\hat{\psi}_\sigma(\mathbf{r},\tau)$ in terms of the fermionic creation ($\hat{\psi}^\dagger$) annihlation ($\hat{\psi}$) operators, respectively.

In the following, we will restrict to the case of non-interacting electrons described by the free Hamiltonian $\hat{H}_0 - \mu\hat{N} = \sum_{\mathbf{k},\sigma} \xi_{\mathbf{k}} \hat{c}^\dagger_{\mathbf{k},\sigma}\hat{c}_{\mathbf{k},\sigma}$, being $\hat{c}^\dagger_{\mathbf{k},\sigma}$ und $\hat{c}_{\mathbf{k},\sigma}$ the creation/annihilation fermionic operators in the momentum represetation.

Within this assumption:

a) perform the Fourier transform of the general expression for $\chi_{nn}(\mathbf{r}-\mathbf{r}',\tau-\tau')$ in Eq. (1), showing that, in momentum space, it reads:

$$\chi_{nn}(\mathbf{q},\tau-\tau') = \left\langle T_\tau\, \hat{n}(\mathbf{q},\tau)\hat{n}(-\mathbf{q},\tau')\right\rangle, \tag{2}$$

where

$$\hat{n}(\mathbf{q},\tau) = \sum_{\mathbf{k},\sigma} \hat{c}^\dagger_{\mathbf{k}+\mathbf{q},\sigma}(\tau)\hat{c}_{\mathbf{k},\sigma}(\tau);$$

[*Hint: One possible way is to perform the generic Fourier trafo w.r.t. the two space variables* **r** *and* **r'** *and, then, to exploit the space translational invariance of the system.*]

b) apply the Wick theorem to Eq. (2) to obtain an explicit expression of $\chi_{nn}(\mathbf{q},\tau-\tau')$ in terms of the non-interacting Green's function $G^0_\sigma(\mathbf{k},\tau-\tau') = -\langle T_\tau \hat{c}_{\mathbf{k},\sigma}(\tau)\hat{c}^\dagger_{\mathbf{k},\sigma}(\tau')\rangle$;

c) draw the graphical (Feynman diagrammatic) representation of the two contributions you will have found by computing the explicit expression of $\chi_{nn}(\mathbf{q},\tau-\tau')$ in **8b)**. What is the physical meaning of the two terms?

d) perform the Fourier transform from (imaginary) times to Matsubara frequencies of the explicit expression for $\chi_{nn}(\mathbf{q},\tau-\tau')$, obtained in **8b)**

[*Hint: Simpler expressions are obtained by explicitly exploiting the (imaginary) time translational invariance of the system.*]

e) Let assume now that the electrons are interacting with a (fully unscreened) Coulomb potential $V(\mathbf{q}) = \frac{e^2}{|\mathbf{q}|^2}$, whose graphical representation is, conventionbally, a wiggle line . How would you draw the lowest order Feynman-diagrammatic corrections to the non-interacting diagrams of **8c)**?

* Bonus points

Viel Spaß!