10. Given is the Lagrangian of the two real scalar fields ϕ_1 , ϕ_2

$$\mathcal{L} = \frac{1}{2} (\partial \phi_1)^2 + \frac{1}{2} (\partial \phi_2)^2 - \frac{\mu^2}{2} \left(\phi_1^2 + \phi_2^2 \right) - \frac{\lambda}{4} \left(\phi_1^2 + \phi_2^2 \right)^2$$
 (15)

with continous SO(2) symmetry

$$\begin{pmatrix} \phi_1' \\ \phi_2' \end{pmatrix} = \begin{pmatrix} \cos \alpha & \sin \alpha \\ -\sin \alpha & \cos \alpha \end{pmatrix} \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix}. \tag{16}$$

Calculate the extreme value of the potential for $\mu^2 < 0$. Does the choice of

$$\langle 0|\phi_1|0\rangle = v, \quad \langle 0|\phi_2|0\rangle = 0,$$
 (17)

for the vaccum expectation value restrict the general solution unjustifiably? Calculate the particle spectrum (the masses of ϕ_1 and ϕ_2) and derive the Feynman rules.

Calculate the infinitesimal matrix of the field transformation for $\alpha \ll 1$ with

$$\phi_i' = \phi_i + i\alpha T_{ij}\phi_i. \tag{18}$$

Calculate the conserved current J_{μ} using

$$J_{\mu} = -i \frac{\partial \mathcal{L}}{\partial (\partial^{\mu} \phi_i)} T_{ij} \phi_j \tag{19}$$

and for the corresponding charge Q(t) using

$$Q = \int J_0 d^3x \,. \tag{20}$$

11. Given is the Lagrangian

$$\mathcal{L} = \frac{v^2}{8} \left(g \, A_{\mu}^3 - g' \, B_{\mu} \right)^2, \tag{21}$$

where A^3_{μ} is the third component of the SU(2) gauge field and B_{μ} is the gauge field of the U(1) symmetry Rewrite it into a quadratic form. Diagonalize the resulting matrix using the orthogonal transformation

$$\begin{pmatrix} Z_{\mu} \\ A_{\mu} \end{pmatrix} = \begin{pmatrix} \cos \theta_{W} & -\sin \theta_{W} \\ \sin \theta_{W} & \cos \theta_{W} \end{pmatrix} \begin{pmatrix} A_{\mu}^{3} \\ B_{\mu} \end{pmatrix}$$
 (22)

and show that the relations

$$\tan \theta_W = \frac{g'}{g},\tag{23}$$

and

$$M_Z^2 = \frac{v^2}{4} \left(g^2 + g'^2 \right) \tag{24}$$

hold.