10. Given is the Lagrangian of the two real scalar fields ϕ_1, ϕ_2

$$\mathcal{L} = \frac{1}{2}(\partial\phi_1)^2 + \frac{1}{2}(\partial\phi_2)^2 - \frac{\mu^2}{2}\left(\phi_1^2 + \phi_2^2\right) - \frac{\lambda}{4}\left(\phi_1^2 + \phi_2^2\right)^2 \tag{15}$$

with continous SO(2) symmetry

$$\begin{pmatrix} \phi_1' \\ \phi_2' \end{pmatrix} = \begin{pmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{pmatrix} \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix}. \tag{16}$$

Calculate the extreme value of the potential for $\mu^2 < 0$. Does the choice of

$$\langle 0|\phi_1|0\rangle = v, \quad \langle 0|\phi_2|0\rangle = 0\,, \tag{17}$$

for the vaccum expectation value restrict the general solution unjustifiably? Calculate the particle spectrum (the masses of ϕ_1 and ϕ_2) and derive the Feynman rules.

Calculate the infinitesimal matrix of the field transformation for $\alpha \ll 1$ with

$$\phi_i' = \phi_i + i\alpha T_{ij}\phi_j\,. \tag{18}$$

Calculate the conserved current J_μ using

$$J_\mu = -i\frac{\partial\mathcal{L}}{\partial(\partial^\mu\phi_i)}\,T_{ij}\phi_j \tag{19}$$

and for the corresponding charge $Q(t)$ using

$$Q = \int J_0 d^3x\,. \tag{20}$$

11. Given is the Lagrangian

$$\mathcal{L} = \frac{v^2}{8}\left(g\,A_\mu^3 - g'\,B_\mu\right)^2, \tag{21}$$

where A_μ^3 is the third component of the SU(2) gauge field and B_μ is the gauge field of the $U(1)$ symmetry Rewrite it into a quadratic form. Diagonalize the resulting matrix using the orthogonal transformation

$$\begin{pmatrix} Z_\mu \\ A_\mu \end{pmatrix} = \begin{pmatrix} \cos\theta_W & -\sin\theta_W \\ \sin\theta_W & \cos\theta_W \end{pmatrix} \begin{pmatrix} A_\mu^3 \\ B_\mu \end{pmatrix} \tag{22}$$

and show that the relations

$$\tan\theta_W = \frac{g'}{g}\,, \tag{23}$$

and

$$M_Z^2 = \frac{v^2}{4}\left(g^2 + g'^2\right) \tag{24}$$

hold.