Complexity Theory

VU 181.142, SS 2014

Homework Assignment 1

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Begin:	18 March, 2014
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Maximum credits:	10

In the lecture "Formale Methoden der Informatik", the following Cook reduction from **co-2-SAT** to the **REACHABILITY** problem was given:

- The variables of φ and their negations form the vertices of $G(\varphi)$.
- There is an arc (α, β) iff there is a clause $\overline{\alpha} \vee \beta$ or $\beta \vee \overline{\alpha}$ in φ, where $\overline{\alpha}$ is the complement of α, i.e.: If α is true in some satisfying assignment $\mathcal{I}$ of φ, then β must also be true in $\mathcal{I}$.
- It can be shown that φ is unsatisfiable iff there is a variable x such that there are paths from x to $\neg x$ and from $\neg x$ to x in $G(\varphi)$.

Notation. It is convenient to write $x \Rightarrow y$ if y is reachable from x in the graph $G(\varphi)$.

Exercise 1 (4 credits) Give a rigorous proof of the "if"-direction of the correctness of the above reduction, i.e.:
If there exists a variable x, s.t. $x \Rightarrow \neg x$ and $\neg x \Rightarrow x$ in $G(\varphi)$, then φ is unsatisfiable.

Hint. Carefully distinguish between what is assumed, what is defined and what has to be shown. The proof could thus start as follows:

Proof. (indirect) Suppose that there exists a variable x, s.t. $x \Rightarrow \neg x$ and $\neg x \Rightarrow x$ in $G(\varphi)$. Moreover, suppose that there exists a model $\mathcal{I}$ of φ. We derive a contradiction by showing that then both x and $\neg x$ are true in $\mathcal{I}$.

For the "only if"-direction of the correctness proof of the problem reduction from **co-2-SAT** to **REACHABILITY**, we have to show the following implication: If there exists no variable x, s.t. $x \Rightarrow \neg x$ and $\neg x \Rightarrow x$ in $G(\varphi)$, then there exists a model $\mathcal{I}$ of φ. To this end, consider the truth assignment $\mathcal{I}$ constructed by the following algorithm:

/* Step 1 */
for each literal x, s.t. $\overline{x} \Rightarrow x$ **do**
    $\mathcal{I}(x) :=$ **true**; /* hence, implicitly, $\mathcal{I}(\overline{x}) :=$ **false**; */

for each literal z with $x \Rightarrow z$ **do** $\mathcal{I}(z) :=$ **true od**;
od;

/* Step 2 */
while there exists $x \in Var(\varphi)$, s.t. $\mathcal{I}(x)$ is undefined **do**
choose an arbitrary variable x, s.t. $\mathcal{I}(x)$ is undefined;
$\mathcal{I}(x) :=$ **true**;
for each literal z with $x \Rightarrow z$ **do** $\mathcal{I}(z) :=$ **true od**;
od;

To show that an assignment $\mathcal{I}$ thus constructed is indeed a model of φ, it is convenient to proof the following lemmas.

Lemma 1 *Suppose that there exists no variable x, s.t. both $x \Rightarrow \overline{x}$ and $\overline{x} \Rightarrow x$ hold. Then a truth assigment made by the above algorithm is never changed later, i.e.: it cannot happen, that at some stage, $\mathcal{I}(z) =$* **true** *for some variable z and later this value is changed to $\mathcal{I}(z) =$* **false** *or vice versa.*

Lemma 2 *Suppose that there exists no variable x, s.t. both $x \Rightarrow \overline{x}$ and $\overline{x} \Rightarrow x$ hold. Then the truth assigment $\mathcal{I}$ constructed by the above algorithm is a model of φ.*

Exercise 2 (4 credits) Give a rigorous proof of Lemma 1.

Exercise 3 (2 credits) Give a rigorous proof of Lemma 2.