Complexity Theory

VU 181.142, SS 2014

Homework Assignment 3

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Begin:	6 May, 2014
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Maximum credits:	10

Exercise 1 (5 credits) Recall the following characterizations of the complexity classes $\Sigma_i \mathsf{P}$ and $\Pi_i \mathsf{P}$ for $i \geq 1$.

Theorem.

- *Let L be a language and $i \geq 1$. Then $L \in \Sigma_i \mathsf{P}$ iff there is a polynomially balanced relation R such that the language $\{x\#y \mid (x, y) \in R\}$ is in $\Pi_{i-1}\mathsf{P}$ and*

$$L = \{x \mid \text{there exists a } y \text{ with } |y| \leq |x|^k \text{ s.t. } (x, y) \in R\}$$

- *Let L be a language and $i \geq 1$. Then $L \in \Pi_i \mathsf{P}$ iff there is a polynomially balanced relation R such that the language $\{x\#y \mid (x, y) \in R\}$ is in $\Sigma_{i-1}\mathsf{P}$ and*

$$L = \{x \mid \text{for all } y \text{ with } |y| \leq |x|^k, (x, y) \in R\}$$

Corollary.

- *Let L be a language and $i \geq 1$. Then $L \in \Sigma_i \mathsf{P}$ iff there is a polynomially balanced, polynomial-time decidable $(i + 1)$-ary relation R such that*

$$L = \{x \mid \exists y_1 \forall y_2 \exists y_3 \cdots Q y_i \text{ such that } (x, y_1, \ldots, y_i) \in R\}$$

  where Q is $\forall$ if i is even and $\exists$ if i is odd.

- *Let L be a language and $i \geq 1$. Then $L \in \Pi_i \mathsf{P}$ iff there is a polynomially balanced, polynomial-time decidable $(i + 1)$-ary relation R such that*

$$L = \{x \mid \forall y_1 \exists y_2 \forall y_3 \cdots Q y_i \text{ such that } (x, y_1, \ldots, y_i) \in R\}$$

  where Q is $\exists$ if i is even and $\forall$ if i is odd.

Give a rigorous proof of this corollary. It suffices to prove the correctness of the characterization of $\Sigma_i \mathsf{P}$. The characterization of $\Pi_i \mathsf{P}$ follows immediately.

Hint. Use the above theorem and proceed by induction on i.

Exercise 2 (5 credits) Recall the $\Sigma_2\mathsf{P}$-hardness proof of **MINIMAL MODEL SAT** by reduction from the QSAT_2-problem: Let an arbitrary instance of QSAT_i be given by the QBF

$$\psi = (\exists x_1, \ldots, x_k)(\forall y_1, \ldots, y_\ell)\varphi$$

Now let $\{x'_1, \ldots, x'_k, z\}$ be fresh propositional variables. Then we construct an instance of **MINIMAL MODEL SAT** by the *variable* z and the *formula*

$$\chi = (\bigwedge_{i=1}^{n} (\neg x_i \leftrightarrow x'_i)) \wedge (\neg\varphi \vee (y_1 \wedge \ldots \wedge y_\ell \wedge z))$$

Recall from the lecture that we have already proved the following implication:
ψ is **true** (in every interpretation) $\Rightarrow$ z is **true** in a minimal model of χ.

Give a rigorous proof also of the opposite direction, i.e.:
z is **true** in a minimal model of χ $\Rightarrow$ ψ is **true** (in every interpretation).

Hint. Let $\mathcal{J}$ be a minimal model of χ and let z be **true** in $\mathcal{J}$.

- First show that then $\mathcal{J}(y_j) =$ **true** for every j.
- Second, let $\mathcal{I}$ be the truth assignment obtained by restricting $\mathcal{J}$ to the variables $\{x_1, \ldots, x_k\}$. Show that (by the minimality of $\mathcal{J}$) $\mathcal{I}$ is indeed a partial assigment on $\{x_1, \ldots, x_k\}$ s.t. for any values assigned to $\{y_1, \ldots, y_\ell\}$, the formula φ is **true**.