Complexity Theory

VU 181.142, SS 2014

Homework Assignment 5

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Exercise 1 (5 credits) Recall the Σ_2P-hardness proof of the Abduction Solvability problem by reduction from QSAT_2: Let an arbitrary instance of the QSAT_2 problem be given by the formula $\varphi = (\exists X)(\forall Y)\psi(X, Y)$ with $X = \{x_1, \ldots, x_k\}$ and $Y = \{y_1, \ldots, y_l\}$. Moreover, let $X' = \{x'_1, \ldots, x'_k\}$, $R = \{r_1, \ldots, r_k\}$, and t be fresh variables. Then we define an instance of Solvability as $\mathcal{P} = \langle V, H, M, T\rangle$ with

$$\begin{aligned} V &= X \cup Y \cup X' \cup R \cup \{t\} \\ H &= X \cup X' \\ M &= R \cup \{t\} \\ T &= \{\psi(X, Y) \to t\} \cup \{\neg x_i \vee \neg x'_i, x_i \to r_i, x'_i \to r_i \mid 1 \leq i \leq k\} \end{aligned}$$

Give a rigorous correctness proof of this problem reduction, i.e., $\varphi \equiv \mathbf{true} \Leftrightarrow Sol(\mathcal{P}) \neq \emptyset$.

Hint. As usual, prove both directions of the equivalence separately. It is convenient to use the notation from the lecture: For $A \subseteq X$, let A' denote the set $\{x' \mid x \in A\}$.

For the "$\Rightarrow$"-direction, you start off with a partial assignment I on X. Let $I^{-1}(\mathbf{true}) = A$. Then it can be shown that $S = A \cup (X \setminus A)'$ is a solution of $\mathcal{P}$. In order to show that S is indeed a solution, you must prove carefully the two conditions that (1) $T \cup S$ is satisfiable and (2) $T \cup S \models M$.

For the "$\Leftarrow$"-direction, first show that a solution S of $\mathcal{P}$ contains exactly one of $\{x_i, x'_i\}$. This is due to the clauses $\{\neg x_i \vee \neg x'_i, x_i \to r_i, x'_i \to r_i\}$ in T. Hence, S must be of the form $S = A \cup (X \setminus A)'$ for some $A \subseteq X$. It remains to show that for the assignment I on X with $I^{-1}(\mathbf{true}) = A$, every extension J of I to the variables Y satisfies the formula $\psi(X, Y)$.

Exercise 2 (5 credits) Recall the Σ_2P-hardness proof of the Abduction Relevance problem by reduction from the Solvability problem: Let an arbitrary instance of the Solvability problem be given by the PAP $\mathcal{P} = \langle V, H, M, T\rangle$. W.l.o.g., let T consist of a single formula

φ and let h, h', m' be fresh variables. Then we define an instance of the Relevance (resp. the Necessity) problem with the following PAP $\mathcal{P}' = \langle V', H', M', T' \rangle$:

$$\begin{aligned} V' &= V \cup \{h, h', m'\} \\ H' &= H \cup \{h, h'\} \\ M' &= M \cup \{m'\} \\ T' &= \{\neg h \vee \varphi\} \cup \{h' \to m \mid m \in M\} \cup \{\neg h \vee \neg h', h \to m', h' \to m'\} \end{aligned}$$

This reduction fulfills the following equivalences:

$\mathcal{P}$ has at least one solution iff h is relevant in $\mathcal{P}'$ iff h' is not necessary in $\mathcal{P}'$.

Give a rigorous proof of these equivalences.

Hints.

- Show both directions of the first equivalence separately:

  For the "$\Rightarrow$"-direction, you start off with a solution S of $\mathcal{P}$ and construct a solution S' of $\mathcal{P}'$ with $h \in S'$. Prove carefully that S' is indeed a solution of $\mathcal{P}'$, i.e. (1) $T' \cup S'$ is satisfiable and (2) $T' \cup S' \models M'$.

  For the "$\Leftarrow$"-direction, you start off with a solution S' of $\mathcal{P}'$, s.t. $h \in S'$ and construct a solution S of $\mathcal{P}$. Prove carefully that S is indeed a solution of $\mathcal{P}$, i.e. (1) $T \cup S$ is satisfiable and (2) $T \cup S \models M$.

- The second equivalence follows easily from the clauses $\{\neg h \vee \neg h', h \to m', h' \to m'\}$ in T', i.e., every solution of $\mathcal{P}'$ contains exactly one of $\{h, h'\}$.