Formal Methods in Computer Science SS 2016:
Optional Exercise Sheet (Proof Structure)

This exercise sheet provides some training material for the optional tutorial. Some of the exercises and solutions have been taken from Wohlgemuth's book.

1.) Translate each of the following statements into a first-order formula using $\in$ as the only predicate symbol:

(a) If $A \subseteq B$, then A and $C \setminus B$ are disjoint.

(b) A is not a subset of B.

(c) $\neg\big(A \cup B \subseteq C \setminus D\big)$.

2.) Eliminate def^2 from the proof fragment below.

1.	$a \in M$	
2.	$M \subseteq N$	
3.	$a \in N$	(1,2; def$^2\subseteq$)

3.) Let A and B be sets. Define $A \setminus B = \{x \mid x \in A \wedge x \notin B\}$ and $\overline{B} = \{x \in U \mid x \notin B\}$, where U is the universal set. Prove $(A \setminus B) = (A \cap \overline{B})$ in the following two steps.

(a) Show: $(A \setminus B) \subseteq (A \cap \overline{B})$.

(b) Show: $(A \cap \overline{B}) \subseteq (A \setminus B)$.

4.) Show: $\forall$ sets A $\forall$ sets B: $((A \setminus B) \cup B) = (A \cup B)$. Contrary to the examples in the lecture, handle the forall statements on the object level and not on the meta level.

5.) Show: $\overline{\bigcup_{i=1}^{n} A_i} = \bigcap_{i=1}^{n} \overline{A_i}$ holds for sets $A_1, \ldots, A_n$.

6.) Assume that A, B, and C are sets. Show: $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$.

7.) For sets A, B, and C:

(a) Show: $A \subseteq B$ iff $A \cap B = A$.

(b) Show: $A \subseteq B$ iff $A \cup B = B$.

8.) Given sets A_k (for $k = 1, 2, 3, 4$), define

$$\begin{aligned} B_1 &= A_1 & \qquad B_2 &= A_2 \setminus A_1 \\ B_3 &= A_3 \setminus (A_1 \cup A_2) & \qquad B_4 &= A_4 \setminus (A_1 \cup A_2 \cup A_3)\,. \end{aligned}$$

(a) Prove: $\bigcup_{i=1}^{4} B_i = \bigcup_{i=1}^{4} A_i$

(b) Prove: For all $1 \le i, j \le 4, i \ne j$: $B_i \cap B_j = \emptyset$

9.) Let $f\colon A \mapsto B$. Suppose $X \subseteq A$, $Y \subseteq B$, and $y \in B$. Define:

P1 $f(X) = \{b \in B \mid b = f(x) \text{ for some } x \in X\}$

P2 $f^{-1}(Y) = \{a \in A \mid f(a) \in Y\}$

P3 $f^{-1}(y) = \{a \in A \mid f(a) = y\}$

Let $C \subseteq B$, and $D \subseteq B$. Show: $f^{-1}(C \cap D) = f^{-1}(C) \cap f^{-1}(D)$. First structure the proof in the same way as we disussed in the lecture. Then present the proof in English.

10.) Prove or refute the following:

Let $f\colon A \mapsto B$, $E \subseteq A$, $F \subseteq A$. Then $f(E \cap F) = f(E) \cap f(F)$.

11.) A function $f\colon A \mapsto B$ is called *one-to-one* provided that for all $a_1, a_2 \in A$: if $f(a_1) = f(a_2)$ then $a_1 = a_2$.

Let $f\colon \mathbb{R} \mapsto \mathbb{R}$ be defined by $f(x) = 2x + 4$ for all $x \in \mathbb{R}$. Prove that f is one-to-one.

12.) A function $f\colon A \mapsto B$ is called *onto* if for each $b \in B$ there exists some $a \in A$ such that $f(a) = b$.

Show that there exists a one-to-one function $g\colon \mathbb{N} \mapsto \mathbb{N}$ which is not onto.

13.) A *sequence* is a function $a\colon \mathbb{N} \mapsto \mathbb{R}$. $a(n)$ is called the nth term of the sequence and is denoted by a_n. The sequence itself is denoted by $\langle a_1, a_2, \ldots \rangle$ or by $\langle a_n \rangle$. A number $L \in \mathbb{R}$ is defined to be the *limit* of $\langle a_n \rangle$, written $\lim_{n\to\infty} \langle a_n \rangle = L$, provided that, given a real number $\epsilon > 0$, there exists a natural number N such that for all $n > N$: $|a_n - L| < \epsilon$. If L is the limit of $\langle a_n \rangle$, then $\langle a_n \rangle$ *converges* to L. A sequence with no limit is said to *diverge*.

Show that $\langle a_n \rangle$ defined by $a_n = n/(n+1)$ has limit 1.

14.) Let $\langle a_n \rangle$ be a sequence. If $\lim_{n\to\infty} \langle a_n \rangle = L$ and $\lim_{n\to\infty} \langle a_n \rangle = M$, then $L = M$. That is, if a sequence has a limit, then it is unique.

Hint: $|x - z| \le |x - y| + |y - z|$ for all $x, y, z \in \mathbb{R}$.